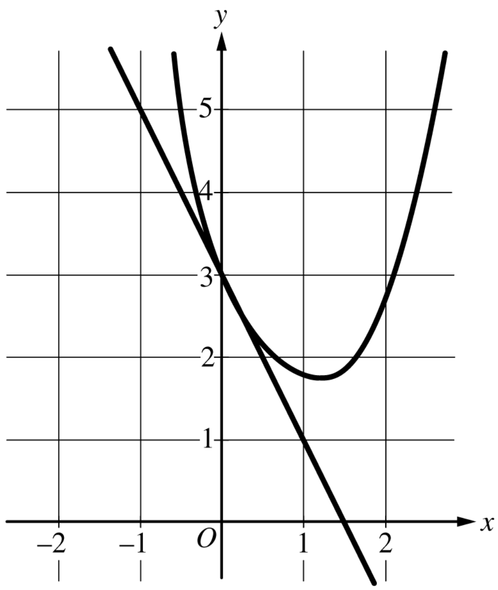
5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

   (a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

   (b) For $k = -8$, find the value of $\int_0^1 f(x)\, dx$.

   (c) For $k = 1$, find the value of $\int_0^2 f(x)\, dx$ or show that it diverges.



n	$f^{(n)}(0)$
2	3
3	$-\frac{23}{2}$
4	54

6. A function f has derivatives of all orders for all real numbers x. A portion of the graph of f is shown above, along with the line tangent to the graph of f at $x = 0$. Selected derivatives of f at $x = 0$ are given in the table above.

(a) Write the third-degree Taylor polynomial for f about $x = 0$.

(b) Write the first three nonzero terms of the Maclaurin series for e^x. Write the second-degree Taylor polynomial for $e^x f(x)$ about $x = 0$.

(c) Let h be the function defined by $h(x) = \int_0^x f(t)\,dt$. Use the Taylor polynomial found in part (a) to find an approximation for $h(1)$.

(d) It is known that the Maclaurin series for h converges to $h(x)$ for all real numbers x. It is also known that the individual terms of the series for $h(1)$ alternate in sign and decrease in absolute value to 0. Use the alternating series error bound to show that the approximation found in part (c) differs from $h(1)$ by at most 0.45.

STOP

END OF EXAM

Question 5

(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2-2x+k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$

$$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$$

(b) $\dfrac{1}{x^2-2x-8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow 1 = A(x+2) + B(x-4)$

$\Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$

$$\begin{aligned} \int_0^1 f(x)\,dx &= \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2}\right) dx \\ &= \left[\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2|\right]_{x=0}^{x=1} \\ &= \left(\frac{1}{6}\ln 3 - \frac{1}{6}\ln 3\right) - \left(\frac{1}{6}\ln 4 - \frac{1}{6}\ln 2\right) = -\frac{1}{6}\ln 2 \end{aligned}$$

$$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$$

(c)
$$\begin{aligned} \int_0^2 \frac{1}{x^2-2x+1}\,dx &= \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx \\ &= \lim_{b\to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx \\ &= \lim_{b\to 1^-} \left(-\frac{1}{x-1}\Big|_{x=0}^{x=b}\right) + \lim_{b\to 1^+} \left(-\frac{1}{x-1}\Big|_{x=b}^{x=2}\right) \\ &= \lim_{b\to 1^-} \left(-\frac{1}{b-1} - 1\right) + \lim_{b\to 1^+} \left(-1 + \frac{1}{b-1}\right) \end{aligned}$$

Because $\lim\limits_{b\to 1^-} \left(-\dfrac{1}{b-1}\right)$ does not exist, the integral diverges.

$$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$$