| $r$ (centimeters) | 0 | 1 | 2 | 2.5 | 4 |
|---|---|---|---|---|---|
| $f(r)$ (milligrams per square centimeter) | 1 | 2 | 6 | 10 | 18 |

1. The density of a bacteria population in a circular petri dish at a distance $r$ centimeters from the center of the dish is given by an increasing, differentiable function $f$, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of $r$ are given in the table above.

(a) Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi \int_0^4 r f(r)\, dr$. Approximate the value of $2\pi \int_0^4 r f(r)\, dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

(c) Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

(d) The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function $g$ defined by $g(r) = 2 - 16(\cos(1.57\sqrt{r}))^3$. For what value of $k$, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$ ?

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**



**GO ON TO THE NEXT PAGE.**

2. For time $t \geq 0$, a particle moves in the $xy$-plane with position $(x(t), y(t))$ and velocity vector $\left\langle (t-1)e^{t^2}, \sin\left(t^{1.25}\right) \right\rangle$. At time $t = 0$, the position of the particle is $(-2, 5)$.

   (a) Find the speed of the particle at time $t = 1.2$. Find the acceleration vector of the particle at time $t = 1.2$.

   (b) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 1.2$.

   (c) Find the coordinates of the point at which the particle is farthest to the left for $t \geq 0$. Explain why there is no point at which the particle is farthest to the right for $t \geq 0$.

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**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**



**GO ON TO THE NEXT PAGE.**

## Part A (AB or BC): Graphing calculator required

## Question 1 — 9 points

**General Scoring Notes**

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

| $r$ (centimeters) | 0 | 1 | 2 | 2.5 | 4 |
|---|---|---|---|---|---|
| $f(r)$ (milligrams per square centimeter) | 1 | 2 | 6 | 10 | 18 |

The density of a bacteria population in a circular petri dish at a distance $r$ centimeters from the center of the dish is given by an increasing, differentiable function $f$, where $f(r)$ is measured in milligrams per square centimeter. Values of $f(r)$ for selected values of $r$ are given in the table above.

| Model Solution | Scoring | |
|---|---|---|
| **(a)** Use the data in the table to estimate $f'(2.25)$. Using correct units, interpret the meaning of your answer in the context of this problem. | | |
| $f'(2.25) \approx \dfrac{f(2.5)-f(2)}{2.5-2} = \dfrac{10-6}{0.5} = 8$ | Estimate | **1 point** |
| At a distance of $r = 2.25$ centimeters from the center of the petri dish, the density of the bacteria population is increasing at a rate of $8$ milligrams per square centimeter per centimeter. | Interpretation with units | **1 point** |

**Scoring notes:**

- To earn the first point the response must provide both a difference and a quotient and must explicitly use values of $f$ from the table.
- Simplification of the numerical value is not required to earn the first point. If the numerical value is simplified, it must be correct.
- The interpretation requires all of the following: distance $r = 2.25$, density of bacteria (population) is increasing or changing, at a rate of $8$, and units of milligrams per square centimeter per centimeter.
- The second point (interpretation) cannot be earned without a nonzero presented value for $f'(2.25)$.
- To earn the second point the interpretation must be consistent with the presented nonzero value for $f'(2.25)$. In particular, if the presented value for $f'(2.25)$ is negative, the interpretation must include "decreasing at a rate of $|f'(2.25)|$" or "changing at a rate of $f'(2.25)$." The second point cannot be earned for an incorrect statement such as "the bacteria density is decreasing at a rate of $-8$ … " even for a presented $f'(2.25) = -8$.
- The units ( $\text{mg/cm}^2/\text{cm}$ ) may be attached to the estimate of $f'(2.25)$ and, if so, do not need to be repeated in the interpretation.
- If units attached to the estimate do not agree with units in the interpretation, read the units in the interpretation.

**Total for part (a) 2 points**

**(b)** The total mass, in milligrams, of bacteria in the petri dish is given by the integral expression $2\pi\int_0^4 r\,f(r)\,dr$. Approximate the value of $2\pi\int_0^4 r\,f(r)\,dr$ using a right Riemann sum with the four subintervals indicated by the data in the table.

| | | |
|---|---|---|
| $2\pi\int_0^4 r\,f(r)\,dr \approx 2\pi(1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1)$ $+ 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5))$ | Right Riemann sum setup | **1 point** |
| $= 2\pi(1\cdot 2\cdot 1 + 2\cdot 6\cdot 1 + 2.5\cdot 10\cdot 0.5 + 4\cdot 18\cdot 1.5)$ $= 269\pi = 845.088$ | Approximation | **1 point** |

**Scoring notes:**

- The presence or absence of $2\pi$ has no bearing on earning the first point.
- The first point is earned for a sum of four products with at most one error in any single value among the four products. Multiplication by $1$ in any term does not need to be shown, but all other products must be explicitly shown.
- A response of $1\cdot f(1)\cdot(1-0) + 2\cdot f(2)\cdot(2-1) + 2.5\cdot f(2.5)\cdot(2.5-2) + 4\cdot f(4)\cdot(4-2.5)$ earns the first point but not the second.
- A response with any error in the Riemann sum is not eligible for the second point.
- A response that provides a completely correct left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ and approximation ($91\pi$) earns one of the two points. A response that has any error in a left Riemann sum or evaluation for $2\pi\int_0^4 r\,f(r)\,dr$ earns no points.
- A response that provides a completely correct right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ and approximation ($80\pi$) earns one of the two points. A response that has any error in a right Riemann sum or evaluation for $2\pi\int_0^4 f(r)\,dr$ earns no points.
- Simplification of the numerical value is not required to earn the second point. If a numerical value is given, it must be correct to three decimal places.

**Total for part (b) 2 points**

**(c)** Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

| | | |
|---|---|---|
| $\frac{d}{dr}(r f(r)) = f(r) + r f'(r)$ | Product rule expression for $\frac{d}{dr}(r f(r))$ | **1 point** |
| Because $f$ is nonnegative and increasing, $\frac{d}{dr}(r f(r)) > 0$ on the interval $0 \le r \le 4$. Thus, the integrand $r f(r)$ is strictly increasing.<br><br>Therefore, the right Riemann sum approximation of $2\pi\int_0^4 r f(r)\, dr$ is an overestimate. | Answer with explanation | **1 point** |

**Scoring notes:**

- To earn the second point a response must explain that $r f(r)$ is increasing and, therefore, the right Riemann sum is an overestimate. The second point can be earned without having earned the first point.
- A response that attempts to explain based on a left Riemann sum for $2\pi\int_0^4 r f(r)\, dr$ from part (b) earns no points.
- A response that attempts to explain based on a right Riemann sum for $2\pi\int_0^4 f(r)\, dr$ from part (b) earns no points.

**Total for part (c) 2 points**

**(d)** The density of bacteria in the petri dish, for $1 \le r \le 4$, is modeled by the function $g$ defined by $g(r) = 2 - 16\left(\cos\left(1.57\sqrt{r}\right)\right)^3$. For what value of $k$, $1 < k < 4$, is $g(k)$ equal to the average value of $g(r)$ on the interval $1 \le r \le 4$?

| | | |
|---|---|---|
| Average value $= g_{\text{avg}} = \dfrac{1}{4-1}\displaystyle\int_1^4 g(r)\,dr$ | Definite integral | **1 point** |
| $\dfrac{1}{4-1}\displaystyle\int_1^4 g(r)\,dr = 9.875795$ | Average value | **1 point** |
| $g(k) = g_{\text{avg}} \Rightarrow k = 2.497$ | Answer | **1 point** |

**Scoring notes:**

- The first point is earned for a definite integral, with or without $\dfrac{1}{4-1}$ or $\dfrac{1}{3}$.
- A response that presents a definite integral with incorrect limits but a correct integrand earns the first point.
- Presentation of the numerical value $9.875795$ is not required to earn the second point. This point can be earned by the average value setup: $\dfrac{1}{3}\displaystyle\int_1^4 g(r)\,dr$.
- Once earned for the average value setup, the second point cannot be lost. Subsequent errors will result in not earning the third point.
- The third point is earned only for the value $k = 2.497$.
- The third point cannot be earned without the second.
- Special case: A response that does not provide the average value setup but presents an average value of $-13.955$ is using degree mode on their calculator. This response would not earn the second point but could earn the third point for an answer of $k = 2.5$ (or $2.499$).

**Total for part (d) 3 points**

**Total for question 1 9 points**

| Model Solution | Scoring | |
|---|---|---|
| **(a)** Find the area of the region in the first quadrant bounded by the $x$-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$. | | |
| $6x\sqrt{4 - x^2} = 0 \Rightarrow x = 0, x = 2$<br><br>$\text{Area} = \int_0^2 6x\sqrt{4 - x^2}\, dx$ | Integrand | **1 point** |
| Let $u = 4 - x^2$.<br><br>$du = -2x\, dx \Rightarrow -\frac{1}{2}\, du = x\, dx$<br>$x = 0 \Rightarrow u = 4 - 0^2 = 4$<br>$x = 2 \Rightarrow u = 4 - 2^2 = 0$<br><br>$\int_0^2 6x\sqrt{4 - x^2}\, dx = \int_4^0 6\left(-\frac{1}{2}\right)\sqrt{u}\, du = -3\int_4^0 u^{1/2}\, du = 3\int_0^4 u^{1/2}\, du$<br>$= 2u^{3/2}\Big\vert_{u=0}^{u=4} = 2 \cdot 8 = 16$ | Antiderivative | **1 point** |
| The area of the region is $16$ square inches. | Answer | **1 point** |

**Scoring notes:**

- Units are not required for any points in this question and are not read if presented (correctly or incorrectly) in any part of the response.
- The first point is earned for presenting $cx\sqrt{4 - x^2}$ or $6x\sqrt{4 - x^2}$ as the integrand in a definite integral. Limits of integration (numeric or alphanumeric) must be presented (as part of the definite integral) but do not need to be correct in order to earn the first point.
- If an indefinite integral is presented with an integrand of the correct form, the first point can be earned if the antiderivative (correct or incorrect) is eventually evaluated using the correct limits of integration.
- The second point can be earned without the first point. The second point is earned for the presentation of a correct antiderivative of a function of the form $Ax\sqrt{4 - x^2}$, for any nonzero constant $A$. If the response has subsequent errors in simplification of the antiderivative or sign errors, the response will earn the second point but will not earn the third point.
- Responses that use $u$-substitution and have incorrect limits of integration or do not change the limits of integration from $x$- to $u$-values are eligible for the second point.
- The response is eligible for the third point only if it has earned the second point.
- The third point is earned only for the answer $16$ or equivalent. In the case where a response only presents an indefinite integral, the use of the correct limits of integration to evaluate the antiderivative must be shown to earn the third point.
- The response cannot correct $-16$ to $+16$ in order to earn the third point; there is no possible reversal here.

**Total for part (a) 3 points**

**(b)** It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is $1.2$ inches. What is the value of $c$ for this spinning toy?

| | | |
|---|---|---|
| The cross-sectional circular slice with the largest radius occurs where $cx\sqrt{4 - x^2}$ has its maximum on the interval $0 < x < 2$. <br><br> $\dfrac{dy}{dx} = \dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0 \Rightarrow x = \sqrt{2}$ | Sets $\dfrac{dy}{dx} = 0$ | **1 point** |
| $x = \sqrt{2} \Rightarrow y = c\sqrt{2}\sqrt{4 - \left(\sqrt{2}\right)^2} = 2c$ <br> $2c = 1.2 \Rightarrow c = 0.6$ | Answer | **1 point** |

**Scoring notes:**

- The first point is earned for setting $\dfrac{dy}{dx} = 0$, $\dfrac{c\left(4 - 2x^2\right)}{\sqrt{4 - x^2}} = 0$, or $c\left(4 - 2x^2\right) = 0$.
- An unsupported $x = \sqrt{2}$ does not earn the first point.
- The second point can be earned without the first point but is earned only for the answer $c = 0.6$ with supporting work.

**Total for part (b)** **2 points**