5. Let $y = f(x)$ be the particular solution to the differential equation $\dfrac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\dfrac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

6. The function g has derivatives of all orders for all real numbers. The Maclaurin series for g is given by

$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2e^n + 3}$ on its interval of convergence.

(a) State the conditions necessary to use the integral test to determine convergence of the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$.

Use the integral test to show that $\sum_{n=0}^{\infty} \frac{1}{e^n}$ converges.

(b) Use the limit comparison test with the series $\sum_{n=0}^{\infty} \frac{1}{e^n}$ to show that the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ converges absolutely.

(c) Determine the radius of convergence of the Maclaurin series for g.

(d) The first two terms of the series $g(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2e^n + 3}$ are used to approximate $g(1)$. Use the alternating series error bound to determine an upper bound on the error of the approximation.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

GO ON TO THE NEXT PAGE.

Part B (BC): Graphing calculator not allowed
Question 5 — 9 points

General Scoring Notes

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.

Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$. It can be shown that $f''(1) = 4$.

Model Solution	Scoring	
(a) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(2)$.		
$f'(1) = \left.\frac{dy}{dx}\right\|_{(x,\,y)=(1,\,4)} = 4 \cdot (1 \ln 1) = 0$ The second-degree Taylor polynomial for f about $x = 1$ is $f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = 4 + 0(x-1) + \frac{4}{2}(x-1)^2.$	Polynomial	**1 point**
$f(2) \approx 4 + 2(2-1)^2 = 6$	Approximation	**1 point**

Scoring notes:

- The first point is earned for $4 + \frac{4 \cdot \ln 1}{1!}(x-1)^1 + \frac{4}{2!}(x-1)^2$ or any correctly simplified equivalent expression. A term involving $(x-1)$ is not necessary. The polynomial must be written about (centered at) $x = 1$.
- If the first point is earned, the second point is earned for just "6" with no additional supporting work.
- If the polynomial is never explicitly written, the first point is not earned. In this case, to earn the second point supporting work of at least "$4 + 2(1)$" is required.

Total for part (a) 2 points

(b) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(2)$. Show the work that leads to your answer.

$f(1.5) \approx f(1) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x,\, y)=(1,\, 4)} = 4 + 0.5 \cdot 0 = 4$ $f(2) \approx f(1.5) + 0.5 \cdot \left.\frac{dy}{dx}\right\|_{(x,\, y)=(1.5,\, 4)}$	Euler's method with two steps	**1 point**
$\approx 4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5) = 4 + 3 \ln 1.5$	Answer	**1 point**

Scoring notes:

- The first point is earned for two steps (of size 0.5) of Euler's method, with at most one error. If there is any error, the second point is not earned.
- To earn the first point a response must contain two Euler steps, $\Delta x = 0.5,$ use of the correct expression for $\frac{dy}{dx}$, and use of the initial condition $f(1) = 4$.
  - The two Euler steps may be explicit expressions or may be presented in a table. Here is a minimal example of a (correctly labeled) table.

x	y	$\Delta y = \frac{dy}{dx} \cdot \Delta x$ or $\Delta y = \frac{dy}{dx} \cdot (0.5)$
1	4	0
1.5	4	$3 \ln 1.5$
2	$4 + 3 \ln 1.5$	

  - Note: In the presence of the correct answer, such a table does not need to be labeled in order to earn both points. In the presence of an incorrect answer, the table must be correctly labeled for the response to earn the first point.
- A single error in computing the approximation of $f(1.5)$ is not considered a second error if that incorrect value is imported correctly into an approximation of $f(2)$.
- Both points are earned for "$4 + 0.5 \cdot 0 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$" or "$4 + 0.5 \cdot 4 \cdot (1.5 \ln 1.5)$".
- Both points are earned for presenting the ordered pair $(2,\, 4 + 3 \ln 1.5)$ with sufficient supporting work.

Total for part (b) **2 points**

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = y \cdot (x \ln x)$ with initial condition $f(1) = 4$.

$\frac{1}{y}\,dy = x\ln x\,dx$	Separation of variables	**1 point**
Using integration by parts, $u = \ln x \quad du = \frac{1}{x}\,dx$ $dv = x\,dx \quad v = \frac{x^2}{2}$ $\int x\ln x\,dx = \frac{x^2}{2}\cdot\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $x\ln x$	**1 point**
$\ln\lvert y\rvert = \frac{x^2\ln x}{2} - \frac{x^2}{4} + C$	Antiderivative for $\frac{1}{y}$	**1 point**
$\ln 4 = 0 - \frac{1}{4} + C \Rightarrow C = \ln 4 + \frac{1}{4}$	Constant of integration and uses initial condition	**1 point**
$y = e^{\left(\frac{x^2\ln x}{2} - \frac{x^2}{4} + \ln 4 + \frac{1}{4}\right)}$ Note: This solution is valid for $x > 0$.	Solves for y	**1 point**

Scoring notes:

- A response with no separation of variables earns 0 out of 5 points. If an error in separation results in one side being correct ($\frac{1}{y}\,dy$ or $x\ln x\,dx$), the response is only eligible to earn the corresponding antiderivative point.
- The third point (antiderivative of $\frac{1}{y}$) can be earned for either $\ln y$ or $\ln\lvert y\rvert$.
- A response with no constant of integration can earn at most 3 out of 5 points.
- A response is eligible for the fourth point if it has earned the first point and at least 1 of the 2 antiderivative points.
- A response earns the fourth point by correctly including the constant of integration in an equation and then replacing x with 1 and y with 4.
- A response is eligible for the fifth point only if it has earned the first 4 points.

Total for part (c) 5 points

Total for question 5 9 points