6. The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\dfrac{1}{10^5}$ of the exact value of $f(0.1)$.

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

GO ON TO THE NEXT PAGE.

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes
The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The function f has derivatives of all orders for all real numbers. It is known that $f(0) = 2$, $f'(0) = 3$, $f''(x) = -f\left(x^2\right)$, and $f'''(x) = -2x \cdot f'\left(x^2\right)$.

Model Solution	Scoring	

(a) Find $f^{(4)}(x)$, the fourth derivative of f with respect to x. Write the fourth-degree Taylor polynomial for f about $x = 0$. Show the work that leads to your answer.

Model Solution	Scoring	
$f^{(4)}(x) = -2 \cdot f'\left(x^2\right) + (-2x) f''\left(x^2\right) \cdot 2x$	Form of product rule	**1 point**
	$f^{(4)}(x)$	**1 point**
$f''(0) = -f(0) = -2$ $f'''(0) = -2(0) \cdot f'(0) = 0$ $f^{(4)}(0) = -2 \cdot f'(0) + 0 \cdot f''(0) \cdot 0 = -2 \cdot 3 + 0 = -6$ The fourth-degree Taylor polynomial for f about $x = 0$ is $T_4(x) = 2 + 3x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-6}{4!}x^4$ $= 2 + 3x - x^2 - \frac{1}{4}x^4$	Two terms of polynomial	**1 point**
	Remaining terms	**1 point**

Scoring notes:

- The first point is earned for a correct fourth derivative or for $f^{(4)}(x) = -2 \cdot f'\left(x^2\right) + (-2x) f''\left(x^2\right)$.
- The second point is earned only for a completely correct expression for $f^{(4)}(x)$.
- A response that earns the first point but not the second may evaluate the presented expression for $f^{(4)}(x)$ at $x = 0$ and use the consistent nonzero value in computing the coefficient of x^4 in the fourth-degree Taylor polynomial.
- A polynomial that includes a nonzero third-degree term, any terms of degree greater than four, or $+\ldots$ does not earn the fourth point.

Total for part (a) — 4 points

(b) The fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate $f(0.1)$. Given that $\left|f^{(5)}(x)\right| \le 15$ for $0 \le x \le 0.5$, use the Lagrange error bound to show that this approximation is within $\frac{1}{10^5}$ of the exact value of $f(0.1)$.

By the Lagrange error bound, $$\left|T_4(0.1) - f(0.1)\right| \le \frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|}{5!} \cdot (0.1)^5 \le \frac{15}{120} \cdot \frac{1}{10^5} \le \frac{1}{10^5}$$	Form of error bound	**1 point**
	Shows $\lvert\text{Error}\rvert \le \frac{1}{10^5}$	**1 point**

Scoring notes:

- The first point is earned for $\frac{\max\limits_{0 \le x \le 0.1}\left|f^{(5)}(x)\right|}{5!} \cdot (0.1)^5$ or $\frac{15}{5!}(0.1)^5$. Subsequent errors in simplification will not earn the second point.
- To earn the second point a response must communicate the inequality $\text{Error} \le \frac{15}{5!} \cdot (0.1)^5 \le \frac{1}{10^5}$.
- A response that states $\text{Error} = \frac{15}{5!} \cdot (0.1)^5$ or $\text{Error} = \frac{1}{10^5}$ does not earn the second point.

Total for part (b) 2 points

(c) Let g be the function such that $g(0) = 4$ and $g'(x) = e^x f(x)$. Write the second-degree Taylor polynomial for g about $x = 0$.

$g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$	$g''(x)$	**1 point**
$g'(0) = e^0 \cdot f(0) = 2$ $g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0) = 2 + 3 = 5$ The second-degree Taylor polynomial for g about $x = 0$ is $T_2(x) = 4 + 2x + \frac{5}{2}x^2$.	First two terms of polynomial	**1 point**
	Taylor polynomial	**1 point**

Scoring notes:

- The first point is earned for $g''(x) = e^x \cdot f(x) + e^x \cdot f'(x)$, $g''(0) = e^0 \cdot f(0) + e^0 \cdot f'(0)$, or $g''(0) = f(0) + f'(0)$.
- A presented polynomial of the form $4 + 2x + ax^2$ earns the second point with or without any supporting work for the first two terms.
- A response that earned neither the first nor the second point only earns the third point for a polynomial of the form $a + bx + \frac{c}{2}x^2$, where $c \ne 0$ is declared to be $g''(0)$.
- A presented polynomial with no support for the coefficient of x^2 does not earn the third point.

- A polynomial that includes any terms of degree greater than two, or $+\ldots,$ does not earn the third point.
- Alternate solution:

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

$$e^x f(x) = \left(1 + x + \frac{x^2}{2} + \cdots\right)\left(2 + 3x - x^2 + \cdots\right) = 2 + 5x + \cdots$$

$$g(x) = \int e^x f(x)\, dx = C + 2x + \frac{5}{2}x^2 + \cdots$$

$$g(0) = 4 \Rightarrow C = 4$$

$$g(x) \approx 4 + 2x + \frac{5}{2}x^2$$

  - A response that is using this alternate solution method earns the first point for $e^x f(x) = 2 + 5x + \cdots,$ the second point for any two correct terms in a second-degree polynomial, and the third point for a completely correct second-degree Taylor polynomial with supporting work.
  - Note: There is not enough information to conclude that $f(x)$ is equal to its Maclaurin series on its interval of convergence. The second and third lines of the alternate solution are being accepted as identifications of the Maclaurin series for $e^x f(x)$ and $g(x),$ respectively.

Total for part (c) 3 points

Total for question 6 9 points