6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

(a) Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $\left| f(-3) - S_3 \right| < \frac{1}{50}$.

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

GO ON TO THE NEXT PAGE.

Part B (BC): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 6^n}$ and converges to $f(x)$ for all x in the interval of convergence. It can be shown that the Maclaurin series for f has a radius of convergence of 6.

	Model Solution	Scoring	
(a)	Determine whether the Maclaurin series for f converges or diverges at $x = 6$. Give a reason for your answer.		
	At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.	Considers $\frac{(n+1)6^n}{n^2 6^n}$	**1 point**
	Because $\frac{n+1}{n^2} > \frac{1}{n}$ for all $n \ge 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by the comparison test.	Answer with reason	**1 point**

Scoring notes:

- To earn the first point using either the comparison or limit comparison test, a response must consider the term $\frac{(n+1)6^n}{n^2 6^n}$. This could be shown by considering the term $\frac{n+1}{n^2}$, either individually or as part of a sum.
- To earn the second point using the comparison test a response must demonstrate that the terms $\frac{n+1}{n^2}$ are larger than the terms in a divergent series.
  - "$\frac{n+1}{n^2} > \frac{1}{n}$, diverges" earns both points.
  - The response does not need to use the term "comparison test," but the response cannot declare use of an incorrect test.
- Alternate solution (limit comparison test):

  At $x = 6$, the series is $\sum_{n=1}^{\infty} \frac{(n+1)6^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$.

  Because $\lim_{n\to\infty} \frac{\frac{n+1}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{n^2} = 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by the limit comparison test.

  - To earn the second point using the limit comparison test, a response must correctly write the limit of the ratio of the terms in the given series to the terms of a divergent series and demonstrate that the limit of this ratio is 1.
  - The response does not need to use the term "limit comparison test," but the response cannot declare use of an incorrect test.

Total for part (a) 2 points

(b) It can be shown that $f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n$ and that the first three terms of this series sum to $S_3 = -\frac{125}{144}$. Show that $\left|f(-3) - S_3\right| < \frac{1}{50}$.

$f(-3) = \sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n^2 6^n} = \sum_{n=1}^{\infty} \frac{(n+1)}{n^2}\left(-\frac{1}{2}\right)^n$ is an alternating series with terms that decrease in magnitude to 0. By the alternating series error bound, $\sum_{n=1}^{3} \frac{n+1}{n^2}\left(-\frac{1}{2}\right)^n = -\frac{125}{144}$ approximates $f(-3)$ with error of at most $\left|\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4\right| = \frac{5}{256} < \frac{5}{250} = \frac{1}{50}$. Thus, $\left|f(-3) - S_3\right| < \frac{1}{50}$.	Uses fourth term Verification	**1 point** **1 point**

Scoring notes:

- The first point is earned for correctly using $x = -3$ in the fourth term. (Listing the fourth term as part of a polynomial is not sufficient.) Using $x = -3$ in any term of degree five or higher does not earn this point.
- The expression $\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4$ earns the first point, but just $\frac{5}{256}$ does not earn the first point.
- A response including the expression $\frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4$ that is subsequently simplified incorrectly earns the first point but not the second.
- To earn the second point the response must state that the series for $f(-3)$ is alternating or that the alternating series error bound is being used.
  - A response of just "Error $\le \frac{4+1}{4^2}\left(-\frac{1}{2}\right)^4 < \frac{1}{50}$" (or any equivalent mathematical expression) earns both points, provided it is accompanied by an indication that the series is alternating.
- A response that declares the error is equal to $\frac{5}{256}$ (or any equivalent form of this value) does not earn the second point.

Total for part (b) 2 points

(c) Find the general term of the Maclaurin series for f', the derivative of f. Find the radius of convergence of the Maclaurin series for f'.

The general term of the Maclaurin series for f' is $\dfrac{(n+1)nx^{n-1}}{n^2 6^n} = \dfrac{(n+1)x^{n-1}}{n \cdot 6^n}$.	General term	**1 point**
Because the radius of convergence of the Maclaurin series for f is 6, the radius of convergence of the Maclaurin series for f' is also 6.	Radius	**1 point**

Scoring notes:

- A response of $\dfrac{(n+1)nx^{n-1}}{n^2 6^n}$ earns the first point. Any expression mathematically equivalent to this also earns the first point.
- The response need not simplify $\dfrac{(n+1)nx^{n-1}}{n^2 6^n}$, but any presented simplification must be correct in order to earn the first point.
- The second point is earned only for a supported answer of 6. The second point can be earned without the first.
- Alternate solution for second point (ratio test):

$$\lim_{n\to\infty}\left|\frac{\dfrac{(n+2)x^n}{(n+1)6^{n+1}}}{\dfrac{(n+1)x^{n-1}}{n\cdot 6^n}}\right| = \lim_{n\to\infty}\left|\frac{n(n+2)}{(n+1)^2}\cdot\frac{x}{6}\right| = \frac{|x|}{6} < 1 \Rightarrow |x| < 6$$

  Therefore, the radius of convergence of the Maclaurin series for f' is 6.

- Alternate solution for second point (root test):

$$\lim_{n\to\infty}\sqrt[n]{\frac{(n+1)|x|^{n-1}}{n\cdot 6^n}} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{1/n}\cdot|x|^{-1/n}\cdot\frac{|x|}{6} = 1\cdot 1\cdot\frac{|x|}{6} < 1 \Rightarrow |x| < 6$$

  Therefore, the radius of convergence of the Maclaurin series for f' is 6.

Total for part (c) 2 points

(d) Let $g(x) = \sum_{n=1}^{\infty} \frac{(n+1)x^{2n}}{n^2 3^n}$. Use the ratio test to determine the radius of convergence of the Maclaurin series for g.

$\left\lvert \dfrac{\dfrac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}{\dfrac{(n+1)x^{2n}}{n^2 3^n}} \right\rvert = \left\lvert \dfrac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \dfrac{n^2 3^n}{(n+1)x^{2n}} \right\rvert$	Sets up ratio	**1 point**
$\lim_{n\to\infty} \left\lvert \dfrac{(n+2)n^2}{(n+1)^3} \right\rvert \cdot \left\lvert \dfrac{x^2}{3} \right\rvert = \left\lvert \dfrac{x^2}{3} \right\rvert$	Limit	**1 point**
$\left\lvert \dfrac{x^2}{3} \right\rvert < 1 \Rightarrow x^2 < 3 \Rightarrow \lvert x \rvert < \sqrt{3}$ The radius of convergence of g is $\sqrt{3}$.	Radius of convergence	**1 point**

Scoring notes:

- The first point is earned by presenting a correct ratio with or without absolute values. Once earned, this point cannot be lost. Any errors in simplification or evaluation of the limit will not earn the second point.
- The first point is earned for ratios mathematically equivalent to any of the following:
  $$\frac{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}},\quad \frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}},\quad \frac{\frac{(n+1)x^{2n}}{n^2 3^n}}{\frac{nx^{2n-2}}{(n-1)^2 3^{n-1}}},\ \text{or}\ \frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n-1)^2 3^{n-1}}{nx^{2n-2}}.$$
- The first point is also earned for ratios mathematically equivalent to the following reciprocal ratios:
  $$\frac{\frac{(n+1)x^{2n}}{n^2 3^n}}{\frac{(n+2)x^{2n+2}}{(n+1)^2 3^{n+1}}},\quad \frac{(n+1)x^{2n}}{n^2 3^n} \cdot \frac{(n+1)^2 3^{n+1}}{(n+2)x^{2n+2}},\quad \frac{\frac{nx^{2n-2}}{(n-1)^2 3^{n-1}}}{\frac{(n+1)x^{2n}}{n^2 3^n}},\ \text{or}\ \frac{nx^{2n-2}}{(n-1)^2 3^{n-1}} \cdot \frac{n^2 3^n}{(n+1)x^{2n}}.$$
  Responses including any of these reciprocal ratios can earn the second point for using limit notation to correctly find a limit of the absolute value of their ratio to be $\left\lvert \frac{3}{x^2} \right\rvert$. Such responses earn the third point only for a final answer of $\sqrt{3}$ with a valid explanation for reporting the reciprocal of $\frac{1}{\sqrt{3}}$.
- To earn the second point a response must use the ratio and correctly evaluate the limit of the ratio, using correct limit notation.
- The third point is earned only for an answer of $\sqrt{3}$ with supporting work.

Total for part (d) 3 points

Total for question 6 9 points