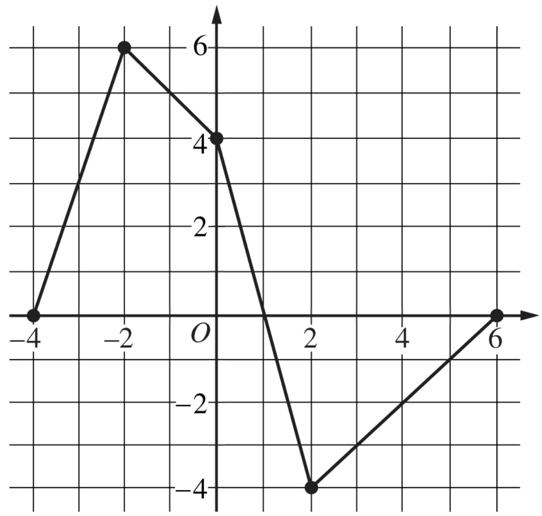


Graph of $f$

4. Let $f$ be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of $f$, consisting of four line segments, is shown above. Let $G$ be the function defined by $G(x) = \int_0^x f(t)\, dt$.

(a) On what open intervals is the graph of $G$ concave up? Give a reason for your answer.

(b) Let $P$ be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

(c) Find $\lim_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of $G$ on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value $c$, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

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5. Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

 (a) Show that $\dfrac{dy}{dx} = \dfrac{y \cos x}{4y - \sin x}$.

 (b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

 (c) For $0 \le x \le \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

 (d) Determine whether $f$ has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

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**Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.**

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**(c)** Is the approximation found in part (b) an overestimate or underestimate of the total mass of bacteria in the petri dish? Explain your reasoning.

| | | |
|---|---|---|
| $\frac{d}{dr}(r\,f(r)) = f(r) + r\,f'(r)$ | Product rule expression for $\frac{d}{dr}(r\,f(r))$ | **1 point** |
| Because $f$ is nonnegative and increasing, $\frac{d}{dr}(r\,f(r)) > 0$ on the interval $0 \le r \le 4$. Thus, the integrand $r\,f(r)$ is strictly increasing.<br><br>Therefore, the right Riemann sum approximation of $2\pi\int_0^4 r\,f(r)\,dr$ is an overestimate. | Answer with explanation | **1 point** |

**Scoring notes:**

- To earn the second point a response must explain that $r\,f(r)$ is increasing and, therefore, the right Riemann sum is an overestimate. The second point can be earned without having earned the first point.
- A response that attempts to explain based on a left Riemann sum for $2\pi\int_0^4 r\,f(r)\,dr$ from part (b) earns no points.
- A response that attempts to explain based on a right Riemann sum for $2\pi\int_0^4 f(r)\,dr$ from part (b) earns no points.

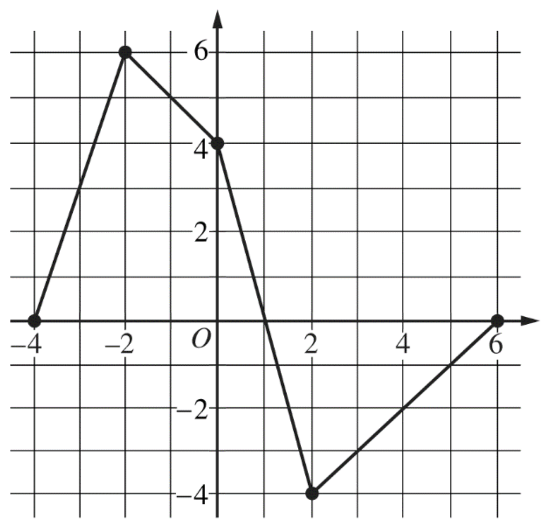
**Total for part (c) 2 points**

## Part B (AB or BC): Graphing calculator not allowed
## Question 4 — 9 points

**General Scoring Notes**

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Scoring guidelines and notes contain examples of the most common approaches seen in student responses. These guidelines can be applied to alternate approaches to ensure that these alternate approaches are scored appropriately.




Graph of $f$

Let $f$ be a continuous function defined on the closed interval $-4 \le x \le 6$. The graph of $f$, consisting of four line segments, is shown above. Let $G$ be the function defined by $G(x) = \int_0^x f(t)\,dt$.

| Model Solution | Scoring | |
|---|---|---|
| $G'(x) = f(x)$ in any part of the response. | $G'(x) = f(x)$ | **1 point** |

**Scoring notes:**

- This "global point" can be earned in any one part. Expressions that show this connection and therefore earn this point include: $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$ in part (a), $G'(3) = f(3)$ in part (b), or $G'(2) = f(2)$ in part (c).

**Total** **1 point**

**(a)** On what open intervals is the graph of $G$ concave up? Give a reason for your answer.

| | | |
|---|---|---|
| $G'(x) = f(x)$<br>The graph of $G$ is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals. | Answer with reason | **1 point** |

**Scoring notes:**

- Intervals may also include one or both endpoints.

**Total for part (a)** **1 point**

**(b)** Let $P$ be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

| | | |
|---|---|---|
| $P'(x) = G(x) \cdot f'(x) + f(x) \cdot G'(x)$<br>$P'(3) = G(3) \cdot f'(3) + f(3) \cdot G'(3)$ | Product rule | **1 point** |
| Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ into the above expression for $P'(3)$ gives the following: | $G(3)$ or $G'(3)$ | **1 point** |
| $P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$ | Answer | **1 point** |

**Scoring notes:**

- The first point is earned for the correct application of the product rule in terms of $x$ or in the evaluation of $P'(3)$. Once earned, this point cannot be lost.
- The second point is earned by correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$.
- To be eligible to earn the third point a response must have earned the first two points.
- Simplification of the numerical value is not required to earn the third point.

**Total for part (b)** **3 points**

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**(c)** Find $\lim_{x\to 2} \frac{G(x)}{x^2 - 2x}$.

| | | |
|---|---|---|
| $\lim_{x\to 2}\left(x^2 - 2x\right) = 0$<br>Because $G$ is continuous for $-4 \le x \le 6$,<br>$\lim_{x\to 2} G(x) = \int_0^2 f(t)\, dt = 0$.<br>Therefore, the limit $\lim_{x\to 2} \frac{G(x)}{x^2 - 2x}$ is an indeterminate form of type $\frac{0}{0}$. | Uses L'Hospital's Rule | **1 point** |
| Using L'Hospital's Rule,<br>$\lim_{x\to 2} \frac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \frac{G'(x)}{2x - 2}$<br>$= \lim_{x\to 2} \frac{f(x)}{2x - 2} = \frac{f(2)}{2} = \frac{-4}{2} = -2$ | Answer with justification | **1 point** |

**Scoring notes:**

- To earn the first point the response must show $\lim_{x\to 2}\left(x^2 - 2x\right) = 0$ and $\lim_{x\to 2} G(x) = 0$ and must show a ratio of the two derivatives, $G'(x)$ and $2x - 2$. The ratio may be shown as evaluations of the derivatives at $x = 2$, such as $\frac{G'(2)}{2}$.
- To earn the second point the response must evaluate correctly with appropriate limit notation. In particular the response must include $\lim_{x\to 2} \frac{G'(x)}{2x - 2}$ or $\lim_{x\to 2} \frac{f(x)}{2x - 2}$.
- With any linkage errors (such as $\frac{G'(x)}{2x - 2} = \frac{f(2)}{2}$), the response does not earn the second point.

**Total for part (c) 2 points**

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**(d)** Find the average rate of change of $G$ on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value $c$, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

| | | |
|---|---|---|
| $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$<br><br>Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$ | Average rate of change | **1 point** |
| Yes, $G'(x) = f(x)$ so $G$ is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$. Therefore, the Mean Value Theorem applies and guarantees a value $c$, $-4 < c < 2$, such that $G'(c) = \dfrac{8}{3}$. | Answer with justification | **1 point** |

**Scoring notes:**

- To earn the first point a response must present at least a difference and a quotient and a correct evaluation. For example, $\dfrac{0+16}{6}$ or $\dfrac{G(2) - G(-4)}{6} = \dfrac{16}{6}$.
- Simplification of the numerical value is not required to earn the first point, but any simplification must be correct.
- The second point can be earned without the first point if the response has the correct setup but an incorrect or no evaluation of the average rate of change. The Mean Value Theorem need not be explicitly stated provided the conditions and conclusion are stated.

**Total for part (d)** **2 points**

**Total for question 4** **9 points**

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