6. Consider the curve given by the equation $6xy = 2 + y^3$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{2y}{y^2 - 2x}$.

(b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\dfrac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\dfrac{dx}{dt} = \dfrac{2}{3}$ unit per second. What is the value of $\dfrac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

Write your responses to this question only on the designated pages in the separate Free Response booklet. Write your solution to each part in the space provided for that part.

GO ON TO THE NEXT PAGE.

Part B (AB): Graphing calculator not allowed
Question 6 — 9 points

General Scoring Notes

The model solution is presented using standard mathematical notation.

Answers (numeric or algebraic) need not be simplified. Answers given as a decimal approximation should be correct to three places after the decimal point. Within each individual free-response question, at most one point is not earned for inappropriate rounding.

Consider the curve given by the equation $6xy = 2 + y^3$.

Model Solution	Scoring	

(a) Show that $\frac{dy}{dx} = \frac{2y}{y^2 - 2x}$.

$\frac{d}{dx}(6xy) = \frac{d}{dx}\left(2 + y^3\right) \Rightarrow 6y + 6x\frac{dy}{dx} = 3y^2\frac{dy}{dx}$	Implicit differentiation	**1 point**
$\Rightarrow 2y = \frac{dy}{dx}\left(y^2 - 2x\right) \Rightarrow \frac{dy}{dx} = \frac{2y}{y^2 - 2x}$	Verification	**1 point**

Scoring notes:

- The first point is earned only for the correct implicit differentiation of $6xy = 2 + y^3$. Responses may use alternative notations for $\frac{dy}{dx}$, such as y'.
- The second point cannot be earned without the first point.
- It is sufficient to present $2y = \frac{dy}{dx}\left(y^2 - 2x\right)$ to earn the second point, provided there are no subsequent errors.

Total for part (a) — 2 points

(b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal, or explain why no such point exists.

For the line tangent to the curve to be horizontal, it is necessary that $2y = 0$ (so $y = 0$) and that $y^2 - 2x \neq 0$.	Sets $2y = 0$	**1 point**
Substituting $y = 0$ into $6xy = 2 + y^3$ yields the equation $6x \cdot 0 = 2$, which has no solution. Therefore, there is no point on the curve at which the line tangent to the curve is horizontal.	Answer with reason	**1 point**

Scoring notes:

- The first point is earned with any of $2y = 0$, $y = 0$, $\frac{dy}{dx} = 0$, $dy = 0$, $y' = 0$, or $\frac{2y}{y^2 - 2x} = 0$.
- A response need not state that at a horizontal tangent, $y^2 - 2x \neq 0$.

Total for part (b) 2 points

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical, or explain why no such point exists.

For a line tangent to this curve to be vertical, it is necessary that $2y \neq 0$ and that $y^2 - 2x = 0$ (so $x = \frac{y^2}{2}$).	Sets $y^2 - 2x = 0$	**1 point**
Substituting $x = \frac{y^2}{2}$ into $6xy = 2 + y^3$ yields the equation $3y^2 \cdot y = 2 + y^3 \Rightarrow 2y^3 = 2 \Rightarrow y = 1$.	Substitutes $x = \frac{y^2}{2}$ into $6xy = 2 + y^3$	**1 point**
Substituting $y = 1$ in $6xy = 2 + y^3$ yields $6x = 2 + 1$, or $x = \frac{1}{2}$. The tangent line to the curve is vertical at the point $\left(\frac{1}{2}, 1\right)$.	Answer	**1 point**

Scoring notes:

- The first point can be earned by presenting $y^2 = 2x$ or $y = \sqrt{2x}$.
- The second point can be earned for the substitution of $y = \sqrt{2x}$ into $6xy = 2 + y^3$, or for substituting $x = \frac{2 + y^3}{6y}$ into $y^2 - 2x = 0$.
- A response earns all three points by setting $y^2 - 2x = 0$, declaring the point $\left(\frac{1}{2}, 1\right)$, and verifying that this point is on the curve $6xy = 2 + y^3$.
- A response that identifies the point $\left(\frac{1}{2}, 1\right)$ but does not verify that the point is on the curve, does not earn the second or the third point.
- To earn the third point the response must present both coordinates of the point $\left(\frac{1}{2}, 1\right)$. The coordinates need not appear as an ordered pair as long as they are labeled.

Total for part (c) 3 points

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ unit per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

$6y\dfrac{dx}{dt} + 6x\dfrac{dy}{dt} = 0 + 3y^2\dfrac{dy}{dt}$	Uses implicit differentiation with respect to t	**1 point**
At the point $(x, y) = \left(\frac{1}{2}, -2\right)$, $6(-2)\left(\frac{2}{3}\right) + 6\left(\frac{1}{2}\right)\dfrac{dy}{dt} = 3(-2)^2\dfrac{dy}{dt}$ $\Rightarrow -8 + 3\dfrac{dy}{dt} = 12\dfrac{dy}{dt}$ $\Rightarrow \dfrac{dy}{dt} = -\dfrac{8}{9}$ unit per second	Answer	**1 point**

Scoring notes:

- The first point is earned by presenting one or more of the terms $6y\dfrac{dx}{dt}$, $6x\dfrac{dy}{dt}$, or $3y^2\dfrac{dy}{dt}$.
- Units will not affect scoring in this part.
- An unsupported response of $-\dfrac{8}{9}$ earns no points.
- Alternate solution:

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\left.\frac{dy}{dx}\right|_{(x,\, y)=(1/2,\, -2)} = \frac{2(-2)}{(-2)^2 - 2(1/2)} = -\frac{4}{3}$$

$$\left.\frac{dy}{dt}\right|_{(x,\, y)=(1/2,\, -2)} = \left.\frac{dy}{dx}\cdot\frac{dx}{dt}\right|_{(x,\, y)=(1/2,\, -2)} = -\frac{4}{3}\cdot\frac{2}{3} = -\frac{8}{9} \text{ unit per second}$$

  - The first point is earned for the statement $\dfrac{dy}{dt} = \dfrac{dy}{dx}\cdot\dfrac{dx}{dt}$ or equivalent.
  - A numerical expression, such as $-\dfrac{4}{3}\cdot\dfrac{2}{3}$ or $\dfrac{2(-2)}{(-2)^2 - 2\left(\frac{1}{2}\right)}\cdot\dfrac{2}{3}$, earns both points.

Total for part (d) **2 points**

Total for question 6 **9 points**