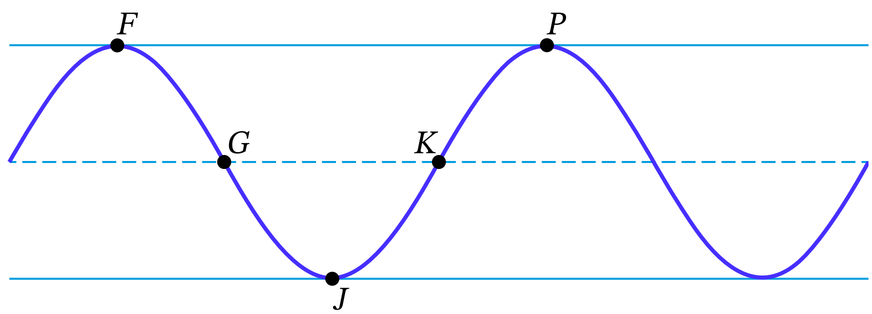
3. For a guitar to make a sound, the strings need to vibrate, or move up and down or back and forth, in a motion that can be modeled by a periodic function.

At time $t = 0$ seconds, point X on one vibrating guitar string starts at its highest position, 2 millimeters above its resting position. Then it passes through its resting position and moves to its lowest position, 2 millimeters below the resting position. Point X then passes through its resting position and returns to 2 millimeters above the resting position. This motion occurs 200 times in 1 second.

The sinusoidal function h models how far point X is from its resting position, in millimeters, as a function of time t, in seconds. A positive value of $h(t)$ indicates the point is above the resting position; a negative value of $h(t)$ indicates the point is below the resting position.

A. The graph of h and its dashed midline for two full cycles is shown. Five points, F, G, J, K, and P, are labeled on the graph. No scale is indicated, and no axes are presented. Determine possible coordinates $(t, h(t))$ for the five points: F, G, J, K, and P.



B. The function h can be written in the form $h(t) = a\sin(b(t+c)) + d$. Find values of constants a, b, c, and d.

C. Refer to the graph of h in part A. The t-coordinate of G is t_1, and the t-coordinate of J is t_2.

i. On the interval (t_1, t_2), which of the following is true about h?

a. h is positive and increasing.

b. h is positive and decreasing.

c. h is negative and increasing.

d. h is negative and decreasing.

ii. On the interval (t_1, t_2), describe the concavity of the graph of h and determine whether the rate of change of h is increasing or decreasing.

4. Directions:

- Unless otherwise specified, the domain of a function *f* is assumed to be the set of all real numbers *x* for which $f(x)$ is a real number. Angle measures for trigonometric functions are assumed to be in radians.
- Solutions to equations must be real numbers. Determine the exact value of any expression that can be obtained without a calculator. For example, $\log_2 8$, $\cos\left(\frac{\pi}{2}\right)$, and $\sin^{-1}(1)$ can be evaluated without a calculator.
- Unless otherwise specified, combine terms using algebraic methods and rules for exponents and logarithms, where applicable. For example, $2x + 3x$, $5^2 \cdot 5^3$, $\frac{x^5}{x^2}$, and $\ln 3 + \ln 5$ should be rewritten in equivalent forms.
- For each part of the question, show the work that leads to your answers.

A. The functions *g* and *h* are given by

$$g(x) = 2\log_3 x$$
$$h(x) = 4\cos^2 x$$

i. Solve $g(x) = 4$ for values of *x* in the domain of *g*.

ii. Solve $h(x) = 3$ for values of *x* in the interval $\left[0, \frac{\pi}{2}\right)$.

B. The functions *j* and *k* are given by

$$j(x) = \log_2 x + 3\log_2 2$$
$$k(x) = \frac{6}{\tan x\left(\csc^2 x - 1\right)}$$

i. Rewrite $j(x)$ as a single logarithm base 2 without negative exponents in any part of the expression. Your result should be of the form $\log_2(\text{expression})$.

ii. Rewrite $k(x)$ as an expression in which $\tan x$ appears exactly once and no other trigonometric functions are involved.

C. The function *m* is given by $m(x) = e^{2x} - e^x - 12$. Find all input values in the domain of *m* that yield an output value of 0.

STOP

END OF EXAM

Question 3: Modeling a Periodic Context
Part B: Graphing calculator not allowed — 6 points

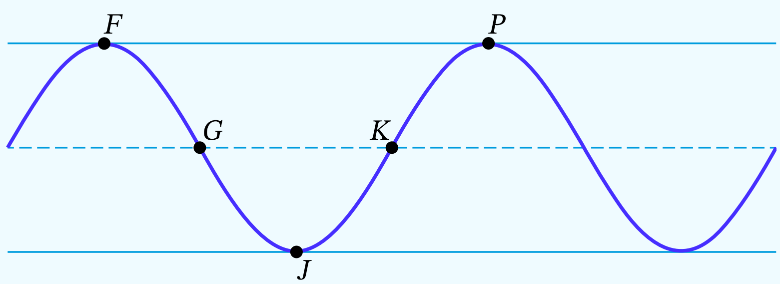
For a guitar to make a sound, the strings need to vibrate, or move up and down or back and forth, in a motion that can be modeled by a periodic function.

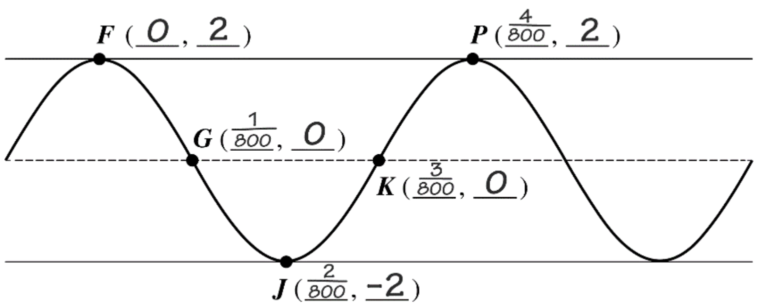
At time $t = 0$ seconds, point X on one vibrating guitar string starts at its highest position, 2 millimeters above its resting position. Then it passes through its resting position and moves to its lowest position, 2 millimeters below the resting position. Point X then passes through its resting position and returns to 2 millimeters above the resting position. This motion occurs 200 times in 1 second.

The sinusoidal function h models how far point X is from its resting position, in millimeters, as a function of time t, in seconds. A positive value of $h(t)$ indicates the point is above the resting position; a negative value of $h(t)$ indicates the point is below the resting position.

	Model Solution	Scoring	

A The graph of h and its dashed midline for two full cycles is shown. Five points, F, G, J, K, and P, are labeled on the graph. No scale is indicated, and no axes are presented.
Determine possible coordinates $(t, h(t))$ for the five points: F, G, J, K, and P.





Note: t-coordinates will vary. A correct set of coordinates for one full cycle of h as pictured is acceptable.

Scoring	Point
$h(t)$-coordinates	**Point A1**
t-coordinates	**Point A2**

Scoring Notes for Part A

- No supporting work is required.
- $h(t)$-coordinates and/or t-coordinates may appear in a list.
- Negative t-coordinates are acceptable. Fractions do not need to be reduced; equivalent fractions and exact decimal values are acceptable.
- t-coordinates must be $0 + \frac{4}{800}k$, $\frac{1}{800} + \frac{4}{800}k$, $\frac{2}{800} + \frac{4}{800}k$, $\frac{3}{800} + \frac{4}{800}k$, $\frac{4}{800} + \frac{4}{800}k$ for a specific integer k.

- If the graph is used to record coordinates, that work is scored. In this case, other work is considered scratchwork and is not scored. Use of the graph is not required.

Partial Credit for Part A

A response that **does not** earn either **Point A1** or **Point A2** is eligible for **partial credit** in part A if the response meets one of the following criteria:

- All 5 points are in the form $(h(t), t)$ with correct input values and correct output values swapped.
- 3 out of the 5 points are correct.
- All 5 points $(t, h(t))$ meet these requirements:
  - t-coordinates are in arithmetic sequence with $\Delta t = \frac{1}{800}$.
  - $h(t)$-coordinates are such that
    1. F and P have **same** $h(t)$-coordinate.
    2. G and K have **same** $h(t)$-coordinate, which is **less than** $h(t)$-coordinate of F and P.
    3. Difference in $h(t)$-coordinates for F and G **equals** Difference in $h(t)$-coordinates for G and J.

Partial credit response is scored **0** for **Point A1** and **1** for **Point A2**.

B The function h can be written in the form $h(t) = a\sin(b(t + c)) + d$. Find values of constants a, b, c, and d.

$\frac{2\pi}{b} = \frac{1}{200}$, so $b = 400\pi$

$d = 0$

Vertical transformations: Values of a and d — **Point B1**

Method 1:

Using $a = 2$,

$c = \frac{1}{800}$ OR $c = \frac{1}{800} + \frac{4}{800}k$, for any integer k

$a = 2$

$b = 400\pi$

$c = \frac{1}{800}$

$d = 0$

For example, $h(t) = 2\sin\left(400\pi\left(t + \frac{1}{800}\right)\right)$. Based on horizontal shifts, there are other correct forms for $h(t)$.

Horizontal transformations: Values of b and c — **Point B2**

Method 2:

Using $a = -2$,

$c = -\frac{1}{800}$ OR $c = -\frac{1}{800} + \frac{4}{800}k$, for any integer k

$a =$	-2
$b =$	400π
$c =$	$-\frac{1}{800}$
$d =$	0

For example, $h(t) = -2\sin\left(400\pi\left(t - \frac{1}{800}\right)\right)$. Based on horizontal shifts, there are other correct forms for $h(t)$.

Scoring Notes for Part B

- No supporting work is required.
- Points are earned for correct values in a list OR for correct values in an expression for $h(t)$. Only one of these answer presentations is required.
- Fractions do not need to be reduced; equivalent fractions and exact decimal values are acceptable.
- If the answer box is used to record values, that work is scored. In this case, other work is considered scratchwork and is not scored. Use of the answer box is not required.
- **Point B1** and **Point B2** may be earned based on the correct use of an imported response from part A that meets these criteria:
  - $a \neq 1$, $b \neq 1$, and $c \neq 0$.
  - All 5 points $(t, h(t))$ from part A meet these requirements:
    - t-coordinates are in arithmetic sequence with $\Delta t = \frac{1}{800}$.
    - $h(t)$-coordinates are such that
      1. F and P have **same** $h(t)$-coordinate.
      2. G and K have **same** $h(t)$-coordinate, which is **less than** $h(t)$-coordinate of F and P.
      3. Difference in $h(t)$-coordinates for F and G **equals** Difference in $h(t)$-coordinates for G and J.

Partial Credit for Part B

A response that **does not** earn either **Point B1** or **Point B2** is eligible for **partial credit** in part B if the response meets one of the following criteria:

- Correct values of a and b [Values of a and b could be $\pm$]
- Correct values of b and d [Value of b could be $\pm$]
- Response uses $h(t) = a\cos(b(t + c)) + d$ with values as follows:
  - $a = 2$; $b = 400\pi$; $c = 0 + \frac{4}{800}k$, for a specific integer k; $d = 0$
  - $a = -2$; $b = 400\pi$; $c = -\frac{2}{800} + \frac{4}{800}k$, for a specific integer k; $d = 0$

Partial credit response is scored **1** for **Point B1** and **0** for **Point B2**.

C Refer to the graph of h in part A. The t-coordinate of G is t_1, and the t-coordinate of J is t_2.

(i) On the interval (t_1, t_2), which of the following is true about h?

a. h is positive and increasing.
b. h is positive and decreasing.
c. h is negative and increasing.
d. h is negative and decreasing.

(ii) On the interval (t_1, t_2), describe the concavity of the graph of h and determine whether the rate of change of h is increasing or decreasing.

(i) Choice d.	Function behavior	**Point C1**
(ii) The graph of h is concave up on the interval (t_1, t_2), and the rate of change of h is increasing on the interval (t_1, t_2).	Concavity of graph and behavior of rate of change	**Point C2**

Scoring Notes for Part C

- No supporting work is required.
- **Point C1** is earned only for a correct answer of "d" OR "negative and decreasing." If both the letter choice and written description are included, the written description is scored.
- To earn **Point C2**, both descriptions must be correct. **Point C2** is not earned for a response that only includes "the graph of h is concave up" OR only includes "the rate of change of h is increasing."
- To earn **Point C2**, "concave up" AND "increasing" is acceptable.
- To earn **Point C2**, "concave up" AND "function h is decreasing at an increasing rate" is acceptable.
- A response with an isolated statement "decreasing at an increasing rate" does not earn **Point C2**. The implied subject is "the rate of change of h."
- A response with a statement that "the rate of change of h is increasing at an increasing (or decreasing) rate" does not earn **Point C2**. Analysis to make such a conclusion requires calculus.
- **Point C2** cannot be earned if there are any errors in part C (ii).