Cambridge A-Level Further Mathematics

FM12 Solutions

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Question 1

(a) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(4r-3)(12r+3) = an^3 + bn^2 + cn$$

where a, b and c are integers to be determined.

Solution

First, expand the expression inside the summation:

$$(4r-3)(12r+3) = 48r^2 + 12r - 36r - 9 = 48r^2 - 24r - 9$$

Using the standard summation formulae from MF19:

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} 1 = n$$

Substitute these into the expanded expression:

$$\begin{aligned}\sum_{r=1}^{n}(48r^2 - 24r - 9) &= 48\left(\frac{n(n+1)(2n+1)}{6}\right) - 24\left(\frac{n(n+1)}{2}\right) - 9n \\ &= 8n(n+1)(2n+1) - 12n(n+1) - 9n \\ &= 8n(2n^2+3n+1) - 12n^2 - 12n - 9n \\ &= 16n^3 + 24n^2 + 8n - 12n^2 - 21n \\ &= 16n^3 + 12n^2 - 13n\end{aligned}$$

Therefore, the integers are $a = 16$, $b = 12$, and $c = -13$.

(b) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(4r-3)(12r+3)}$ in terms of n.

Solution

First, express the term in partial fractions:

$$\frac{1}{(4r-3)(12r+3)} = \frac{1}{3(4r-3)(4r+1)}$$

Let $\frac{1}{(4r-3)(4r+1)} = \frac{A}{4r-3} + \frac{B}{4r+1}$. Solving gives $A = \frac{1}{4}, B = -\frac{1}{4}$. So, the general term can be written as:

$$\frac{1}{12}\left(\frac{1}{4r-3} - \frac{1}{4r+1}\right)$$

Applying the method of differences:

$$\begin{aligned}
r = 1: &\quad \frac{1}{12}\left(1 - \frac{1}{5}\right)\\
r = 2: &\quad \frac{1}{12}\left(\frac{1}{5} - \frac{1}{9}\right)\\
&\vdots\\
r = n: &\quad \frac{1}{12}\left(\frac{1}{4n-3} - \frac{1}{4n+1}\right)
\end{aligned}$$

Summing these equations, the intermediate terms cancel out (telescoping sum):

$$S_n = \frac{1}{12}\left(1 - \frac{1}{4n+1}\right) = \frac{n}{3(4n+1)}$$

(c) State the value of $\sum_{r=1}^{\infty} \frac{1}{(4r-3)(12r+3)}$.

Solution

As $n \to \infty$, the term $\frac{1}{4n+1} \to 0$. Therefore, the value of the infinite series is $\frac{1}{12}$.

Question 2

Prove by mathematical induction that $3^{6n} + 92^n + 89$ is divisible by 91 for all positive integers n.

Solution

Let $f(n) = 3^{6n} + 92^n + 89$.
Step 1: Base case $n = 1$ When $n = 1$:

$$f(1) = 3^6 + 92^1 + 89 = 729 + 92 + 89 = 910$$

Since $910 = 10 \times 91$, $f(1)$ is divisible by 91. The statement holds for $n = 1$.
Step 2: Assumption Assume the statement is true for some positive integer k, meaning $f(k) = 3^{6k} + 92^k + 89$ is divisible by 91 (i.e., $f(k) \equiv 0 \pmod{91}$).
Step 3: Inductive step $n = k + 1$ Consider $f(k + 1)$:

$$\begin{aligned} f(k+1) &= 3^{6(k+1)} + 92^{k+1} + 89 \\ &= 3^{6k} \cdot 3^6 + 92^k \cdot 92 + 89 \\ &= 729 \cdot 3^{6k} + 92 \cdot 92^k + 89 \end{aligned}$$

We rewrite this to include the expression for $f(k)$:

$$\begin{aligned} f(k+1) &= 729(3^{6k} + 92^k + 89) - 729 \cdot 92^k - 729 \cdot 89 + 92 \cdot 92^k + 89 \\ &= 729f(k) - 637 \cdot 92^k - 64792 \end{aligned}$$

Check the remaining terms for divisibility by 91: - $637 \div 91 = 7$ - $64792 \div 91 = 712$ Thus, we can factor out 91:

$$f(k+1) = 729f(k) - 91(7 \cdot 92^k + 712)$$

Since $f(k)$ is divisible by 91 by our assumption, and the second term is clearly a multiple of 91, $f(k + 1)$ must also be divisible by 91.
Conclusion By mathematical induction, $3^{6n} + 92^n + 89$ is divisible by 91 for all positive integers n.

Question 3

The matrix M represents the sequence of two transformations in the $x-y$ plane given by a stretch parallel to the x-axis, scale factor $k (k \neq 0, 1)$, followed by a shear, x-axis fixed, with $(0, 1)$ mapped to $(2, 1)$.

(a) Show that $M = \begin{pmatrix} k & 2 \\ 0 & 1 \end{pmatrix}$.

Solution

Let T_1 be the matrix for a stretch parallel to the x-axis with scale factor k:

$$T_1 = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$$

Let T_2 be the matrix for the shear with the x-axis fixed. The general form is $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. Since $(0, 1)$ is mapped to $(2, 1)$:

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \implies a = 2$$

So, $T_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The combined transformation matrix M is T_2T_1 (applied right to left):

$$M = T_2T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 2 \\ 0 & 1 \end{pmatrix}$$

(b) The transformation represented by M has a line of invariant points. Find, in terms of k, the equation of this line.

Solution

Invariant points satisfy $M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$:

$$\begin{pmatrix} k & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Expanding this gives the system of equations: 1) $kx + 2y = x \implies (k-1)x + 2y = 0$ 2) $y = y$ (always true) Since $k \neq 1$, we can solve for y to find the equation of the line:

$$y = \frac{1-k}{2}x$$

(c) The square S in the $x-y$ plane is transformed by M onto the parallelogram P. Given that the area of S is half of the area of P, find the possible values of k.

Solution

The area scale factor is given by the absolute value of the determinant of matrix M, $|\det(M)|$. Given that $\text{Area}(P) = 2\text{Area}(S)$, the scale factor is 2. Calculate the determinant:

$$\det(M) = (k)(1) - (2)(0) = k$$

Therefore, $|k| = 2$. The possible values are $k = 2$ or $k = -2$.

Question 4

The cubic equation $5x^3 - 2x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$.

Solution

Introduce a new variable $y = \frac{1}{x^2}$, which means $x^2 = \frac{1}{y} \implies x = y^{-1/2}$. Substitute this into the original equation $5x^3 - 2x + 1 = 0$:

$$5(y^{-1/2})^3 - 2(y^{-1/2}) + 1 = 0 \implies 5y^{-3/2} - 2y^{-1/2} + 1 = 0$$

Multiply the entire equation by $y^{3/2}$:

$$5 - 2y + y^{3/2} = 0 \implies y^{3/2} = 2y - 5$$

Square both sides to eliminate the fractional power:

$$y^3 = (2y-5)^2 \implies y^3 = 4y^2 - 20y + 25$$

Rearrange to get the new cubic equation:

$$y^3 - 4y^2 + 20y - 25 = 0$$

(b) Find the value of $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4}$.

Solution

Let the roots of the new equation be Y_1, Y_2, Y_3. The value we need to find is exactly $\sum Y_i^2$. Using Vieta's formulas on the new equation $y^3 - 4y^2 + 20y - 25 = 0$:

$$\sum Y_i = 4, \quad \sum Y_iY_j = 20, \quad Y_1Y_2Y_3 = 25$$

Using the identity $\sum Y_i^2 = (\sum Y_i)^2 - 2\sum Y_iY_j$:

$$\sum Y_i^2 = (4)^2 - 2(20) = 16 - 40 = -24$$

So, $\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} = -24$.

(c) Find also the values of $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}$ and $\frac{1}{\alpha^8} + \frac{1}{\beta^8} + \frac{1}{\gamma^8}$.

Solution

From the equation $y^3 = 4y^2 - 20y + 25$, we can define the sums of powers of roots as $S_n = \sum Y_i^n$. For $n = 3$ (which corresponds to the sum of the inverse 6th powers of the original roots):

$$S_3 = 4S_2 - 20S_1 + 25S_0$$

We know $S_2 = -24, S_1 = 4$, and $S_0 = 3$ (since there are 3 roots):

$$S_3 = 4(-24) - 20(4) + 25(3) = -96 - 80 + 75 = -101$$

Thus, $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6} = -101$.

Multiply the relationship by y to get $y^4 = 4y^3 - 20y^2 + 25y$:

$$S_4 = 4S_3 - 20S_2 + 25S_1$$

Substitute the known values:

$$S_4 = 4(-101) - 20(-24) + 25(4) = -404 + 480 + 100 = 176$$

Thus, $\frac{1}{\alpha^8} + \frac{1}{\beta^8} + \frac{1}{\gamma^8} = 176$.

Question 5

The lines l_1 and l_2 have equations $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = -3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \mu(2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2.

Solution

The direction vectors of the lines are $\mathbf{d}_1 = (3, -4, 2)$ and $\mathbf{d}_2 = (2, 4, 3)$. Find a vector $\mathbf{n}$ perpendicular to both lines using the cross product:

$$\mathbf{n} = \mathbf{d}_1 \times \mathbf{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 2 \\ 2 & 4 & 3 \end{vmatrix} = (-12 - 8)\mathbf{i} - (9 - 4)\mathbf{j} + (12 + 8)\mathbf{k} = (-20, -5, 20)$$

For simpler calculation, we can use a parallel vector $\mathbf{n}' = (4, 1, -4)$. Its magnitude is $|\mathbf{n}'| = \sqrt{16 + 1 + 16} = \sqrt{33}$. Points on the lines are $\mathbf{a}_1 = (-2, 4, 4)$ and $\mathbf{a}_2 = (-3, -2, 4)$. The vector connecting them is $\mathbf{v} = \mathbf{a}_1 - \mathbf{a}_2 = (1, 6, 0)$. The shortest distance is the magnitude of the projection of $\mathbf{v}$ onto $\mathbf{n}'$:

$$D = \frac{|\mathbf{v} \cdot \mathbf{n}'|}{|\mathbf{n}'|} = \frac{|(1)(4) + (6)(1) + (0)(-4)|}{\sqrt{33}} = \frac{10}{\sqrt{33}}$$

(b) The plane Π_1 contains l_1 and the point $P(1, -1, 1)$. The plane Π_2 contains l_2 and passes through the origin. Find the acute angle between Π_1 and Π_2.

Solution

Find the normal vector $\mathbf{n}_1$ for Π_1: The plane contains l_1 and points $A_1(-2, 4, 4)$ and $P(1, -1, 1)$. Vector $A_1P = (1 - (-2), -1 - 4, 1 - 4) = (3, -5, -3)$. $\mathbf{n}_1 = \mathbf{d}_1 \times A_1P = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 2 \\ 3 & -5 & -3 \end{vmatrix} = (12 + 10, -(-9 - 6), -15 + 12) = (22, 15, -3)$.

Find the normal vector $\mathbf{n}_2$ for Π_2: The plane contains l_2 and points $A_2(-3, -2, 4)$ and the origin $O(0, 0, 0)$. Vector $A_2O = (3, 2, -4)$. $\mathbf{n}_2 = \mathbf{d}_2 \times A_2O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 3 \\ 3 & 2 & -4 \end{vmatrix} = (-16 - 6, -(-8 - 9), 4 - 12) = (-22, 17, -8)$.

The acute angle between the planes is given by:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$$

Dot product: $\mathbf{n}_1 \cdot \mathbf{n}_2 = 22(-22) + 15(17) + (-3)(-8) = -484 + 255 + 24 = -205$. Magnitudes: $|\mathbf{n}_1| = \sqrt{22^2 + 15^2 + (-3)^2} = \sqrt{484 + 225 + 9} = \sqrt{718}$ $|\mathbf{n}_2| = \sqrt{(-22)^2 + 17^2 + (-8)^2} = \sqrt{484 + 289 + 64} = \sqrt{837}$

$$\cos\theta = \frac{205}{\sqrt{718}\sqrt{837}} \implies \theta = \arccos\left(\frac{205}{\sqrt{600966}}\right) \approx 74.7^\circ$$

The acute angle is 74.7°.

Question 6

The curve C has equation $y = \frac{x^2-x+2}{4x+4}$.

(a) Find the equations of the asymptotes of C.

Solution

Vertical asymptote: Set the denominator to zero, $4x + 4 = 0 \implies x = -1$.
Oblique asymptote: Perform polynomial long division.

$$x^2 - x + 2 = (4x + 4)\left(\frac{1}{4}x - \frac{1}{2}\right) + 4$$

So, $y = \frac{1}{4}x - \frac{1}{2} + \frac{1}{x+1}$. As $x \to \pm\infty$, the remainder tends to 0, giving the oblique asymptote:

$$y = \frac{1}{4}x - \frac{1}{2}$$

(b) Find the coordinates of any stationary points on C.

Solution

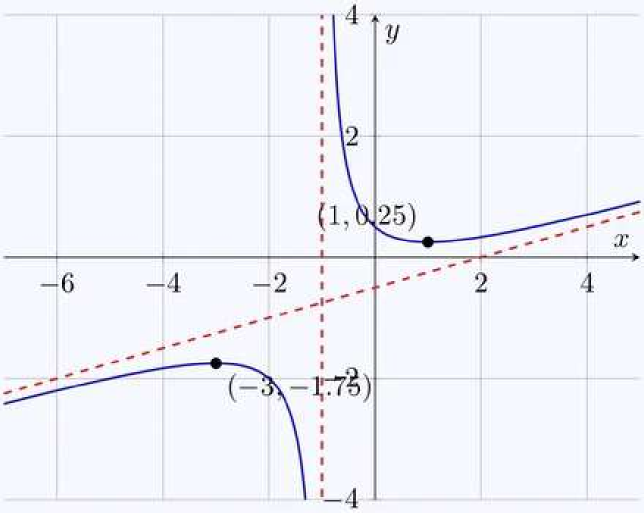
Differentiate and set $y' = 0$:

$$\begin{aligned} y' &= \frac{(2x-1)(4x+4) - (x^2-x+2)(4)}{(4x+4)^2} \\ &= \frac{8x^2 + 8x - 4x - 4 - 4x^2 + 4x - 8}{(4x+4)^2} \\ &= \frac{4x^2 + 8x - 12}{(4x+4)^2} = \frac{4(x+3)(x-1)}{(4x+4)^2} \end{aligned}$$

Setting $y' = 0$ gives $x = 1$ or $x = -3$. Substitute back to find y: When $x = 1$, $y = \frac{1-1+2}{8} = \frac{1}{4}$. When $x = -3$, $y = \frac{9-(-3)+2}{4(-3)+4} = \frac{14}{-8} = -\frac{7}{4}$. The stationary points are $(1, \frac{1}{4})$ and $(-3, -\frac{7}{4})$.

(c) Sketch C.

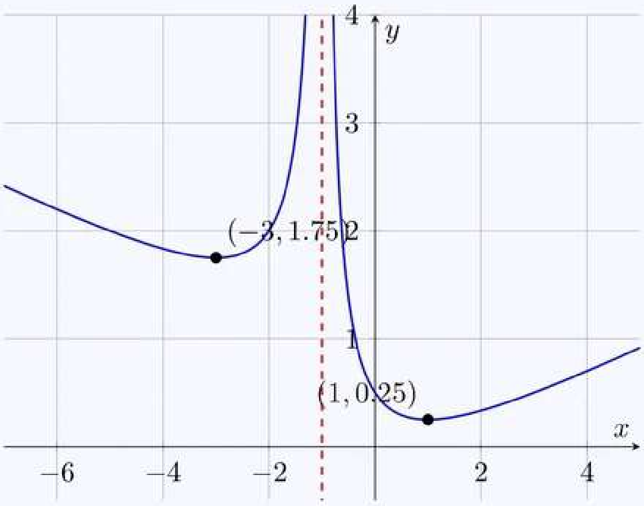
Solution



(d) Sketch the curve with equation $y = \left|\frac{x^2-x+2}{4x+4}\right|$.

Solution

Reflect the part of the curve from part (c) that lies below the x-axis (which is the region $x < -1$) upwards. The stationary point $(-3, -1.75)$ becomes a local minimum at $(-3, 1.75)$. The oblique asymptote for $x < -1$ becomes $y = -0.25x + 0.5$.



(e) Find in exact form the set of values of x for which $\left|\frac{x^2-x+2}{4x+4}\right| < 2$.

Solution

Remove the modulus to solve $-2 < \frac{x^2-x+2}{4x+4} < 2$.
Case 1: $x > -1$, meaning $4x + 4 > 0$ The inequality becomes: $-8x - 8 < x^2 - x + 2 < 8x+8$. Left side: $x^2+7x+10 > 0 \implies (x+2)(x+5) > 0$. This is always true since $x > -1$. Right side: $x^2 - 9x - 6 < 0$. The roots are $x = \frac{9\pm\sqrt{105}}{2}$. Since $\frac{9-\sqrt{105}}{2} \approx -0.62 > -1$, the solution for this region is:

$$\frac{9-\sqrt{105}}{2} < x < \frac{9+\sqrt{105}}{2}$$

Case 2: $x < -1$, meaning $4x + 4 < 0$ Multiplying reverses the inequality: $-8x - 8 > x^2-x+2 > 8x+8$. Left side: $x^2+7x+10 < 0 \implies (x+2)(x+5) < 0 \implies -5 < x < -2$. Right side: $x^2 - 9x - 6 > 0 \implies x < \frac{9-\sqrt{105}}{2}$ or $x > \frac{9+\sqrt{105}}{2}$. This is always true in the interval $-5 < x < -2$. The solution for this region is:

$$-5 < x < -2$$

Final Result: The set of values is the union of both regions:

$$\{x \in \mathbb{R} \mid -5 < x < -2 \text{ or } \frac{9-\sqrt{105}}{2} < x < \frac{9+\sqrt{105}}{2}\}$$

Question 7

The curve C has polar equation $r^2 = \frac{a}{1+\tan 2\theta}$ for $0 \leq \theta < \frac{1}{4}\pi$, where a is a positive constant.

(a) Show that, at the point on C furthest from the initial line, $\cos\theta - \frac{\sin\theta}{\cos 2\theta(\sin 2\theta+\cos 2\theta)} = 0$ and verify that this equation has a root between 0.5 and 0.6.

Solution

The point furthest from the initial line occurs when the y-coordinate is maximized. $y = r\sin\theta \implies y^2 = r^2\sin^2\theta = \frac{a\sin^2\theta}{1+\tan 2\theta}$. Differentiate y^2 with respect to θ and set to 0:

$$\frac{d(y^2)}{d\theta} = a\frac{2\sin\theta\cos\theta(1+\tan 2\theta) - \sin^2\theta(2\sec^2 2\theta)}{(1+\tan 2\theta)^2} = 0$$

The numerator must be zero:

$$2\sin\theta\cos\theta\left(1 + \frac{\sin 2\theta}{\cos 2\theta}\right) - \frac{2\sin^2\theta}{\cos^2 2\theta} = 0$$

Divide by $2\sin\theta$ (since $\theta \neq 0$):

$$\cos\theta\left(\frac{\cos 2\theta + \sin 2\theta}{\cos 2\theta}\right) - \frac{\sin\theta}{\cos^2 2\theta} = 0$$

Divide by $\frac{\cos 2\theta+\sin 2\theta}{\cos 2\theta}$:

$$\cos\theta - \frac{\sin\theta}{\cos 2\theta(\cos 2\theta + \sin 2\theta)} = 0$$

This completes the proof.
To verify the root, let $g(\theta) = \cos\theta - \frac{\sin\theta}{\cos 2\theta(\sin 2\theta+\cos 2\theta)}$. Calculate values at the boundaries (using radians):

$$g(0.5) \approx \cos(0.5) - \frac{\sin(0.5)}{\cos(1)(\sin 1 + \cos 1)} \approx 0.235 > 0$$

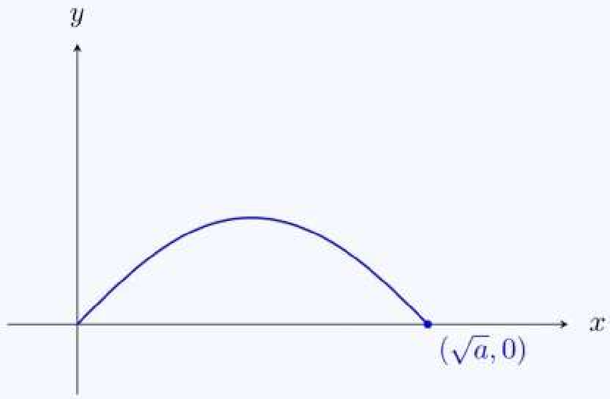
$$g(0.6) \approx \cos(0.6) - \frac{\sin(0.6)}{\cos(1.2)(\sin 1.2 + \cos 1.2)} \approx -0.379 < 0$$

Since there is a sign change between $g(0.5)$ and $g(0.6)$ and the function is continuous on this interval, a root exists between 0.5 and 0.6.

(b) Sketch C.

Solution

When $\theta = 0$, $r = \sqrt{a}$. As $\theta \to \frac{\pi}{4}$, $\tan 2\theta \to \infty$, so $r \to 0$. The curve forms a closed loop in the first quadrant, starting from $(\sqrt{a}, 0)$ and curving into the pole.


y
x
$(\sqrt{a}, 0)$

(c) Use the substitution $v = \tan 2\theta$ to show that the area enclosed by C and the initial line is $\frac{1}{4}a\int_0^\infty \frac{1}{(1+v)(1+v^2)}dv$ and find the exact value of this area.

Solution

The area in polar coordinates is given by Area $= \frac{1}{2}\int_0^{\pi/4} r^2 d\theta$. Substitute r^2:

$$\text{Area} = \frac{1}{2}\int_0^{\pi/4} \frac{a}{1+\tan 2\theta} d\theta$$

Using the substitution $v = \tan 2\theta$:

$$\frac{dv}{d\theta} = 2\sec^2 2\theta = 2(1+\tan^2 2\theta) = 2(1+v^2) \implies d\theta = \frac{dv}{2(1+v^2)}$$

Limits of integration: when $\theta = 0$, $v = 0$; as $\theta \to \frac{\pi}{4}$, $v \to \infty$. Substitute these into the integral:

$$\text{Area} = \frac{1}{2}\int_0^\infty \frac{a}{1+v}\frac{dv}{2(1+v^2)} = \frac{1}{4}a\int_0^\infty \frac{1}{(1+v)(1+v^2)}dv$$

Next, evaluate the integral using partial fractions:

$$\frac{1}{(1+v)(1+v^2)} = \frac{A}{1+v} + \frac{Bv+C}{1+v^2}$$

Multiplying by the denominator: $A(1+v^2) + (Bv+C)(1+v) = 1$. Let $v = -1 \implies 2A = 1 \implies A = \frac{1}{2}$. Let $v = 0 \implies A + C = 1 \implies C = \frac{1}{2}$. Compare coefficients of $v^2 \implies A + B = 0 \implies B = -\frac{1}{2}$. The integral becomes:

$$\begin{aligned}
\frac{1}{4}a\int_0^\infty \left(\frac{1/2}{1+v} + \frac{-1/2v+1/2}{1+v^2}\right)dv &= \frac{a}{8}\int_0^\infty \left(\frac{1}{1+v} - \frac{v}{1+v^2} + \frac{1}{1+v^2}\right)dv \\
&= \frac{a}{8}\left[\ln(1+v) - \frac{1}{2}\ln(1+v^2) + \arctan v\right]_0^\infty \\
&= \frac{a}{8}\left[\ln\left(\frac{1+v}{\sqrt{1+v^2}}\right) + \arctan v\right]_0^\infty
\end{aligned}$$

Evaluate at the limits: As $v \to \infty$, $\ln\left(\frac{v}{\sqrt{v^2}}\right) \to \ln 1 = 0$, and $\arctan v \to \frac{\pi}{2}$. At $v = 0$: $\ln 1 + \arctan 0 = 0$. The exact area is:

$$\text{Area} = \frac{a}{8}\left(0 + \frac{\pi}{2} - 0\right) = \frac{\pi a}{16}$$