A-Level Further Mathematics (9231/22/M/J/26) Worked Solutions

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Question 1

Find the first three terms in the Maclaurin's series for $\sinh^{-1}(x+1)$ in the form $\ln a + bx + cx^2$, giving the exact values of the constants a, b and c. **[5]**

Let $y = f(x) = \sinh^{-1}(x+1)$. We want to find $f(0)$, $f'(0)$, and $f''(0)$.
First, evaluate $f(0)$:

$$f(0) = \sinh^{-1}(1) = \ln(1 + \sqrt{1^2+1}) = \ln(1+\sqrt{2})$$

To find the derivatives, we can use the standard derivative of the inverse hyperbolic sine function:

$$f'(x) = \frac{1}{\sqrt{(x+1)^2+1}} = (x^2+2x+2)^{-\frac{1}{2}}$$

Evaluate $f'(0)$:

$$f'(0) = (0+0+2)^{-\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Differentiate again to find $f''(x)$ using the chain rule:

$$f''(x) = -\frac{1}{2}(x^2+2x+2)^{-\frac{3}{2}} \cdot (2x+2) = -(x+1)(x^2+2x+2)^{-\frac{3}{2}}$$

Evaluate $f''(0)$:

$$f''(0) = -(1)(2)^{-\frac{3}{2}} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$$

The Maclaurin series expansion is given by $f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$:

$$f(x) \approx \ln(1+\sqrt{2}) + \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{8}x^2$$

Comparing this to $\ln a + bx + cx^2$, we find the exact constants to be:

$$a = 1+\sqrt{2}, \quad b = \frac{\sqrt{2}}{2}, \quad c = -\frac{\sqrt{2}}{8}$$

Question 2

Let $I_n = \displaystyle\int_0^1 (1+t)^n \sin \pi t \, dt$, where n is a non-negative integer.

(a) Show that, for $n \geq 2$, $\pi^2 I_n = \pi(2^n+1) - n(n-1)I_{n-2}$. **[4]**

(b) Find the exact value of I_2. **[3]**

Part (a):
We apply integration by parts to I_n. Let $u = (1+t)^n$ and $dv = \sin(\pi t)dt$.
Then $du = n(1+t)^{n-1}dt$ and $v = -\frac{1}{\pi}\cos(\pi t)$.

$$I_n = \left[-\frac{(1+t)^n}{\pi}\cos(\pi t)\right]_0^1 - \int_0^1 \left(-\frac{1}{\pi}\cos(\pi t)\right) n(1+t)^{n-1}dt$$

Evaluating the boundary terms:

$$\left[-\frac{(1+t)^n}{\pi}\cos(\pi t)\right]_0^1 = \left(-\frac{2^n}{\pi}\cos\pi\right) - \left(-\frac{1^n}{\pi}\cos 0\right) = \frac{2^n}{\pi} + \frac{1}{\pi} = \frac{2^n+1}{\pi}$$

So, $I_n = \frac{2^n+1}{\pi} + \frac{n}{\pi}\int_0^1 (1+t)^{n-1}\cos(\pi t)dt$.
Apply integration by parts a second time to the new integral.
Let $u = (1+t)^{n-1}$ and $dv = \cos(\pi t)dt$.
Then $du = (n-1)(1+t)^{n-2}dt$ and $v = \frac{1}{\pi}\sin(\pi t)$.

$$\int_0^1 (1+t)^{n-1}\cos(\pi t)dt = \left[\frac{(1+t)^{n-1}}{\pi}\sin(\pi t)\right]_0^1 - \frac{n-1}{\pi}\int_0^1 (1+t)^{n-2}\sin(\pi t)dt$$

Since $\sin(\pi) = 0$ and $\sin(0) = 0$, the boundary term vanishes. Recognizing the remaining integral as I_{n-2}, we have:

$$\int_0^1 (1+t)^{n-1}\cos(\pi t)dt = 0 - \frac{n-1}{\pi}I_{n-2}$$

Substitute this back into the expression for I_n:

$$I_n = \frac{2^n+1}{\pi} + \frac{n}{\pi}\left(-\frac{n-1}{\pi}I_{n-2}\right) = \frac{2^n+1}{\pi} - \frac{n(n-1)}{\pi^2}I_{n-2}$$

Multiplying the entire equation by π^2 yields the required result:

$$\pi^2 I_n = \pi(2^n+1) - n(n-1)I_{n-2}$$

Part (b):
We are looking for I_2. Using the recurrence relation with $n = 2$:

$$\pi^2 I_2 = \pi(2^2+1) - 2(1)I_0 = 5\pi - 2I_0$$

First, evaluate I_0:

$$I_0 = \int_0^1 (1+t)^0\sin(\pi t)dt = \int_0^1 \sin(\pi t)dt = \left[-\frac{1}{\pi}\cos(\pi t)\right]_0^1 = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$$

Substitute I_0 back into the equation:

$$\pi^2 I_2 = 5\pi - 2\left(\frac{2}{\pi}\right) \implies \pi^2 I_2 = 5\pi - \frac{4}{\pi} \implies I_2 = \frac{5}{\pi} - \frac{4}{\pi^3}$$

Therefore, the particular solution is:

$$y \sinh x = \frac{1}{3} \cosh^3 x - \cosh x + \frac{10}{81}$$

Rearranging to the form $y = f(x)$:

$$y = \frac{27 \cosh^3 x - 81 \cosh x + 10}{81 \sinh x}$$

Question 3

(a) Using the substitution $u = \cosh x$, find $\displaystyle\int \sinh^3 x\, dx$. **[4]**

(b) Find the particular solution of the differential equation

$$\frac{dy}{dx} + y \coth x = \sinh^2 x,$$

given that $y = 0$ when $x = \ln \frac{1}{3}$. Give your answer in the form $y = f(x)$. **[7]**

Solution

Part (a):
Using the substitution $u = \cosh x$, we have $du = \sinh x\, dx$.
Rewrite the integral by separating one $\sinh x$ term:

$$\int \sinh^3 x\, dx = \int \sinh^2 x \cdot \sinh x\, dx = \int (\cosh^2 x - 1) \sinh x\, dx$$

Substituting u and du:

$$\int (u^2 - 1)\, du = \frac{1}{3}u^3 - u + C$$

Replacing u with $\cosh x$ gives:

$$\int \sinh^3 x\, dx = \frac{1}{3} \cosh^3 x - \cosh x + C$$

Part (b):
The differential equation is a first-order linear ODE. The integrating factor is:

$$I.F. = \exp\left(\int \coth x\, dx\right) = \exp(\ln(\sinh x)) = \sinh x$$

Multiplying both sides of the differential equation by $\sinh x$:

$$\sinh x \frac{dy}{dx} + y \cosh x = \sinh^3 x \implies \frac{d}{dx}(y \sinh x) = \sinh^3 x$$

Integrating both sides with respect to x, and using the result from part (a):

$$y \sinh x = \int \sinh^3 x\, dx = \frac{1}{3} \cosh^3 x - \cosh x + C$$

To find the constant C, use the initial condition $y = 0$ when $x = \ln(1/3)$.
First, evaluate $\sinh(\ln(1/3))$ and $\cosh(\ln(1/3))$:

$$\sinh(\ln(1/3)) = \frac{e^{\ln(1/3)} - e^{-\ln(1/3)}}{2} = \frac{1/3 - 3}{2} = -\frac{4}{3}$$

$$\cosh(\ln(1/3)) = \frac{e^{\ln(1/3)} + e^{-\ln(1/3)}}{2} = \frac{1/3 + 3}{2} = \frac{5}{3}$$

Substitute $x = \ln(1/3)$ and $y = 0$:

$$0 \cdot \left(-\frac{4}{3}\right) = \frac{1}{3}\left(\frac{5}{3}\right)^3 - \frac{5}{3} + C \implies 0 = \frac{125}{81} - \frac{135}{81} + C \implies C = \frac{10}{81}$$

Question 4

The curve C has equation $(x-y)^5 + \ln(xy) = 1 + \ln 2$.

(a) Show that, at the point $(-1,-2)$ on C, $\dfrac{dy}{dx} = \dfrac{8}{11}$. **[3]**

(b) Find the value of $\dfrac{d^2y}{dx^2}$ at the point $(-1,-2)$. **[5]**

Part (a):
Differentiate the equation implicitly with respect to x:

$$5(x-y)^4\left(1-\frac{dy}{dx}\right)+\frac{1}{xy}\left(y+x\frac{dy}{dx}\right)=0$$

Substitute the point $x=-1$ and $y=-2$. Note that $x-y=-1-(-2)=1$ and $xy=2$:

$$5(1)^4\left(1-\frac{dy}{dx}\right)+\frac{1}{2}\left(-2-\frac{dy}{dx}\right)=0$$

Expand and solve for $\frac{dy}{dx}$:

$$5-5\frac{dy}{dx}-1-\frac{1}{2}\frac{dy}{dx}=0 \implies 4=\frac{11}{2}\frac{dy}{dx} \implies \frac{dy}{dx}=\frac{8}{11}$$

Part (b):
To find the second derivative, it is easier to rewrite the first implicit derivative expression slightly before differentiating again:

$$5(x-y)^4(1-y')+\frac{1}{x}+\frac{y'}{y}=0$$

Differentiating implicitly with respect to x again:

$$20(x-y)^3(1-y')^2+5(x-y)^4(-y'')-x^{-2}+\frac{y''y-(y')^2}{y^2}=0$$

Substitute the known values: $x=-1, y=-2, (x-y)=1, y'=\frac{8}{11}$:

$$20(1)^3\left(1-\frac{8}{11}\right)^2-5(1)^4y''-\frac{1}{(-1)^2}+\frac{y''(-2)-(\frac{8}{11})^2}{(-2)^2}=0$$

Simplify the terms:

$$20\left(\frac{3}{11}\right)^2-5y''-1+\frac{-2y''-\frac{64}{121}}{4}=0$$

$$\frac{180}{121}-5y''-1-\frac{1}{2}y''-\frac{16}{121}=0$$

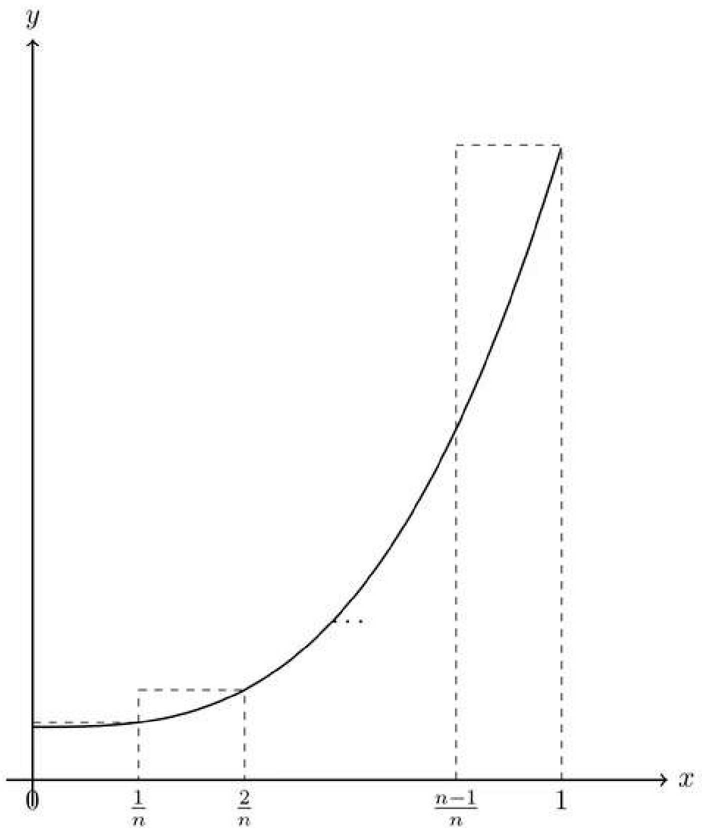
Group the y'' terms and the constant terms:

$$\frac{180-121-16}{121}=\frac{11}{2}y'' \implies \frac{43}{121}=\frac{11}{2}y''$$

Solving for y'':

$$y''=\frac{43\times 2}{121\times 11}=\frac{86}{1331}$$

Question 5



The diagram shows the curve with equation $y = \frac{11}{10}x^3 + \frac{1}{10}$ for $0 \le x \le 1$ together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\displaystyle\int_0^1 \left(\frac{11}{10}x^3 + \frac{1}{10}\right) dx < U_n$ where $U_n = \frac{11}{40}\left(1 + \frac{1}{n}\right)^2 + \frac{1}{10}$. **[4]**

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\displaystyle\int_0^1 \left(\frac{11}{10}x^3 + \frac{1}{10}\right) dx$. **[4]**

(c) Show that $\lim_{n\to\infty}(U_n - L_n) = 0$. **[2]**

Solution

Part (a):
The upper rectangles evaluate the function at the right-endpoints $x_r = \frac{r}{n}$ for $r = 1, 2, \ldots, n$. The total area of these right-endpoint rectangles is:

$$U_n = \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{r}{n}\right) = \sum_{r=1}^{n} \frac{1}{n}\left[\frac{11}{10}\left(\frac{r}{n}\right)^3 + \frac{1}{10}\right]$$

Distributing the sum:

$$U_n = \frac{11}{10n^4}\sum_{r=1}^{n} r^3 + \frac{1}{10n}\sum_{r=1}^{n} 1$$

Using the standard summation formulas $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$ and $\sum_{r=1}^{n} 1 = n$:

$$U_n = \frac{11}{10n^4}\left(\frac{n^2(n+1)^2}{4}\right) + \frac{n}{10n} = \frac{11(n+1)^2}{40n^2} + \frac{1}{10} = \frac{11}{40}\left(1 + \frac{1}{n}\right)^2 + \frac{1}{10}$$

Since the function $y = \frac{11}{10}x^3 + \frac{1}{10}$ is strictly increasing on $[0, 1]$, the right-endpoint rectangles consistently overestimate the true area under the curve. Thus, $\int_0^1 (\frac{11}{10}x^3 + \frac{1}{10})dx < U_n$.

Part (b):
To find a lower bound L_n, we use left-endpoint rectangles. We evaluate the function at $x_r = \frac{r}{n}$ for $r = 0, 1, \ldots, n-1$:

$$L_n = \sum_{r=0}^{n-1} \frac{1}{n}\left[\frac{11}{10}\left(\frac{r}{n}\right)^3 + \frac{1}{10}\right] = \frac{11}{10n^4}\sum_{r=0}^{n-1} r^3 + \frac{1}{10n}\sum_{r=0}^{n-1} 1$$

The sum of the first $n-1$ cubes is $\frac{(n-1)^2n^2}{4}$, and $\sum_{r=0}^{n-1} 1 = n$.

$$L_n = \frac{11}{10n^4}\left(\frac{(n-1)^2n^2}{4}\right) + \frac{1}{10} = \frac{11(n-1)^2}{40n^2} + \frac{1}{10} = \frac{11}{40}\left(1 - \frac{1}{n}\right)^2 + \frac{1}{10}$$

Part (c):
We evaluate the limit of the difference between the upper and lower bounds:

$$U_n - L_n = \left[\frac{11}{40}\left(1 + \frac{1}{n}\right)^2 + \frac{1}{10}\right] - \left[\frac{11}{40}\left(1 - \frac{1}{n}\right)^2 + \frac{1}{10}\right]$$

$$= \frac{11}{40}\left[\left(1 + \frac{1}{n}\right)^2 - \left(1 - \frac{1}{n}\right)^2\right] = \frac{11}{40}\left(\frac{4}{n}\right) = \frac{11}{10n}$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty}(U_n - L_n) = \lim_{n\to\infty}\frac{11}{10n} = 0$$

Question 6

Find the particular solution of the differential equation

$$5\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = x^2 + 7x + 1,$$

given that, when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$. **[10]**

First, find the complementary function (C.F.) by solving the homogeneous equation:

$$5m^2 + 4m + 1 = 0 \implies m = \frac{-4 \pm \sqrt{16-20}}{10} = -\frac{2}{5} \pm \frac{1}{5}i$$

The complementary function is:

$$y_c = e^{-0.4x}\left(A\cos(0.2x) + B\sin(0.2x)\right)$$

Next, find the particular integral (P.I.). Let $y_p = ax^2+bx+c$. Differentiating yields $y_p' = 2ax+b$ and $y_p'' = 2a$. Substitute these into the original differential equation:

$$5(2a) + 4(2ax+b) + (ax^2+bx+c) \equiv x^2 + 7x + 1$$

Grouping the coefficients of x^2, x, and the constants:

$$ax^2 + (8a+b)x + (10a+4b+c) \equiv x^2 + 7x + 1$$

Equating coefficients:

$$\begin{aligned} x^2 &: \quad a = 1 \\ x^1 &: \quad 8(1) + b = 7 \implies b = -1 \\ x^0 &: \quad 10(1) + 4(-1) + c = 1 \implies 6 + c = 1 \implies c = -5 \end{aligned}$$

So the particular integral is $y_p = x^2 - x - 5$. The general solution is therefore:

$$y = e^{-0.4x}\left(A\cos(0.2x) + B\sin(0.2x)\right) + x^2 - x - 5$$

Apply the initial conditions to find A and B. When $x = 0$, $y = 0$:

$$0 = e^0(A\cos 0 + B\sin 0) + 0 - 0 - 5 \implies A - 5 = 0 \implies A = 5$$

To use the second condition, we must differentiate the general solution:

$$\frac{dy}{dx} = -0.4e^{-0.4x}\left(5\cos(0.2x) + B\sin(0.2x)\right) + e^{-0.4x}\left(-1\sin(0.2x) + 0.2B\cos(0.2x)\right) + 2x - 1$$

Substitute $x = 0$ and $\frac{dy}{dx} = 0$:

$$0 = -0.4(5) + 1(0.2B) - 1 \implies 0 = -2 + 0.2B - 1 \implies 0.2B = 3 \implies B = 15$$

The particular solution is:

$$y = e^{-0.4x}\left(5\cos(0.2x) + 15\sin(0.2x)\right) + x^2 - x - 5$$

Question 7

(a) State the sum of the series $z + z^2 + z^3 + \cdots + z^n$ for $z \neq 1$. **[1]**

(b) Find all roots of the equation $1 + z + z^2 + \cdots + z^6 = 0$ in the form $e^{iq\pi}$, where q is rational. **[2]**

(c) Given instead that $z = e^{-1+i\theta}$, use de Moivre's theorem to show that

$$\sum_{m=1}^{\infty} e^{-m} \cos m\theta = \frac{e\cos\theta - 1}{e^2 - 2e\cos\theta + 1}.$$

[7]

Part (a):
This is a finite geometric progression with first term $a = z$ and common ratio $r = z$. The sum is:

$$S_n = \frac{z(1 - z^n)}{1 - z}$$

Part (b):
Notice that $1 + z + z^2 + \cdots + z^6 = \frac{1-z^7}{1-z} = 0$. Since $z \neq 1$, this implies $z^7 = 1$. The roots are the 7th roots of unity, excluding $z = 1$. In exponential form:

$$z = e^{i\frac{2k\pi}{7}}, \quad \text{for } k = 1, 2, 3, 4, 5, 6$$

Part (c):
Consider the infinite geometric series $\sum_{m=1}^{\infty} z^m$. For $|z| < 1$, the sum to infinity is $S_\infty = \frac{z}{1-z}$. Given $z = e^{-1+i\theta} = e^{-1}(\cos\theta + i\sin\theta)$, note that $|z| = e^{-1} < 1$, so the series converges. By De Moivre's theorem, $z^m = e^{-m}(\cos m\theta + i\sin m\theta)$. Summing these gives:

$$\sum_{m=1}^{\infty} z^m = \sum_{m=1}^{\infty} e^{-m}\cos m\theta + i\sum_{m=1}^{\infty} e^{-m}\sin m\theta$$

We need to find the real part of S_∞. Using algebraic manipulation:

$$S_\infty = \frac{z}{1-z} = \frac{e^{-1+i\theta}}{1 - e^{-1+i\theta}}$$

Multiply numerator and denominator by e:

$$S_\infty = \frac{e^{i\theta}}{e - e^{i\theta}} = \frac{\cos\theta + i\sin\theta}{(e - \cos\theta) - i\sin\theta}$$

To realize the denominator, multiply the numerator and denominator by the complex conjugate $(e - \cos\theta) + i\sin\theta$:

$$S_\infty = \frac{(\cos\theta + i\sin\theta)((e - \cos\theta) + i\sin\theta)}{(e - \cos\theta)^2 + \sin^2\theta}$$

Expanding the denominator:

$$(e - \cos\theta)^2 + \sin^2\theta = e^2 - 2e\cos\theta + \cos^2\theta + \sin^2\theta = e^2 - 2e\cos\theta + 1$$

Expanding the real part of the numerator:

$$\text{Re}(\text{Numerator}) = \cos\theta(e - \cos\theta) - \sin\theta(\sin\theta) = e\cos\theta - (\cos^2\theta + \sin^2\theta) = e\cos\theta - 1$$

Thus, equating the real parts of both sides, we obtain the required identity:

$$\sum_{m=1}^{\infty} e^{-m}\cos m\theta = \frac{e\cos\theta - 1}{e^2 - 2e\cos\theta + 1}$$

Question 8

(a) The constants a and b are such that the system of equations

$$\begin{aligned} -x - 2y + z &= 1 \\ ax + y - z &= 2 \\ by + 2z &= 3 \end{aligned}$$

does not have a unique solution. Find a in terms of b. **[3]**

(b) The matrix A is given by

$$A = \begin{pmatrix} -1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Use the characteristic equation of A to show that $A^4 = pA^2 + qI$, where p and q are integers to be determined. **[4]**

(c) Find a matrix P and a diagonal matrix D such that $A^7 = PDP^{-1}$. **[7]**

Solution

Part (a):
For the system to not have a unique solution, the determinant of the coefficient matrix must be zero.

$$\Delta = \begin{vmatrix} -1 & -2 & 1 \\ a & 1 & -1 \\ 0 & b & 2 \end{vmatrix} = 0$$

Expanding along the first column:

$$-1\begin{vmatrix} 1 & -1 \\ b & 2 \end{vmatrix} - a\begin{vmatrix} -2 & 1 \\ b & 2 \end{vmatrix} + 0 = 0$$

$$-1(2 - (-b)) - a(-4 - b) = 0 \implies -2 - b + 4a + ab = 0$$

Rearranging to solve for a:

$$a(4 + b) = b + 2 \implies a = \frac{b+2}{b+4} \quad (\text{for } b \neq -4)$$

Part (b):
Since matrix A is upper triangular, its eigenvalues are simply the entries on the main diagonal: $\lambda = -1, 1, 2$.
The characteristic equation is therefore:

$$(\lambda + 1)(\lambda - 1)(\lambda - 2) = 0 \implies (\lambda^2 - 1)(\lambda - 2) = 0 \implies \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

By the Cayley-Hamilton theorem, the matrix A satisfies its own characteristic equation:

$$A^3 - 2A^2 - A + 2I = \mathbf{0} \implies A^3 = 2A^2 + A - 2I$$

Multiply the entire equation by A to get an expression for A^4:

$$A^4 = 2A^3 + A^2 - 2A$$

Substitute the expression for A^3 into this new equation:

$$A^4 = 2(2A^2 + A - 2I) + A^2 - 2A = 4A^2 + 2A - 4I + A^2 - 2A = 5A^2 - 4I$$

Comparing this to $A^4 = pA^2 + qI$, we find that integers $p = 5$ and $q = -4$.

Part (c):
To find P and D, we need the eigenvectors corresponding to the eigenvalues $-1, 1, 2$.
For $\lambda_1 = -1$, solve $(A - (-1)I)\mathbf{v} = 0$:

$$\begin{pmatrix} 0 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies 3z = 0 \implies z = 0. \text{ Then } -2y = 0 \implies y = 0.$$

Let $x = 1$. Thus, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For $\lambda_2 = 1$, solve $(A - I)\mathbf{v} = 0$:

$$\begin{pmatrix} -2 & -2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies z = 0. \text{ Then } -2x - 2y = 0 \implies y = -x.$$

Let $x = 1$. Thus, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = 2$, solve $(A - 2I)\mathbf{v} = 0$:

$$\begin{pmatrix} -3 & -2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies -y - z = 0 \implies y = -z.$$

Substituting $y = -z$ into the first equation: $-3x - 2(-z) + z = 0 \implies -3x + 3z = 0 \implies x = z$.
Let $z = 1$. Thus, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Matrix P is formed by the eigenvectors as columns, and D is the diagonal matrix of the eigenvalues raised to the 7th power:

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} (-1)^7 & 0 & 0 \\ 0 & 1^7 & 0 \\ 0 & 0 & 2^7 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 128 \end{pmatrix}$$