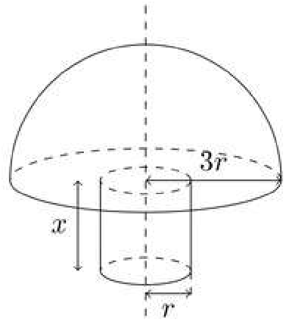
Further Mathematics 32 - Mechanics

Capybara

Question 1

An object consists of a uniform solid hemisphere of radius $3r$ attached to a uniform solid cylinder of radius r and height x. The hemisphere and the cylinder are made of the same material. The axes of symmetry of the hemisphere and the cylinder coincide (see diagram).



a. Find, in terms of r and x, the distance of the centre of mass of the object from the base of the cylinder. [4]

b. The point A lies on the circumference of the base of the hemisphere. When the object is freely suspended from point A, the base of the hemisphere makes an angle θ with the downward vertical through A. It is given that $x = r$. Find the value of θ. [3]

Solution

a. Let the density of the uniform material be ρ. We establish a 1D coordinate system along the axis of symmetry, with the origin $y = 0$ at the flat base of the cylinder.
For the cylinder: Volume $V_c = \pi r^2 x$. Mass $M_c = \rho\pi r^2 x$. Its centre of mass is located at its geometric centre: $y_c = \frac{x}{2}$.
For the hemisphere: Volume $V_h = \frac{2}{3}\pi(3r)^3 = 18\pi r^3$. Mass $M_h = 18\rho\pi r^3$. The centre of mass of a hemisphere of radius R from its flat face is $\frac{3}{8}R$. Here, $R = 3r$, so the distance from the hemisphere's base is $\frac{3}{8}(3r) = \frac{9}{8}r$. Since the hemisphere's base sits on top of the cylinder at $y = x$, its absolute coordinate is $y_h = x + \frac{9}{8}r$.
Composite Object: Taking moments about the base of the cylinder:

$$\bar{y} = \frac{M_c y_c + M_h y_h}{M_c + M_h} = \frac{(\pi r^2 x)(\frac{x}{2}) + (18\pi r^3)(x + \frac{9}{8}r)}{\pi r^2 x + 18\pi r^3}$$

Dividing numerator and denominator by πr^2:

$$\bar{y} = \frac{\frac{1}{2}x^2 + 18rx + \frac{81}{4}r^2}{x + 18r} = \boldsymbol{\frac{2x^2 + 72rx + 81r^2}{4(x + 18r)}}$$

b. Given $x = r$, substitute into the result from (a):

$$\bar{y} = \frac{2(r)^2 + 72r(r) + 81r^2}{4(r + 18r)} = \frac{155r^2}{4(19r)} = \frac{155}{76}r$$

This is the distance from the base of the cylinder. The vertical distance from the *base of the* *hemisphere* to the centre of mass is:

$$d_y = \bar{y} - x = \frac{155}{76}r - r = \frac{79}{76}r$$

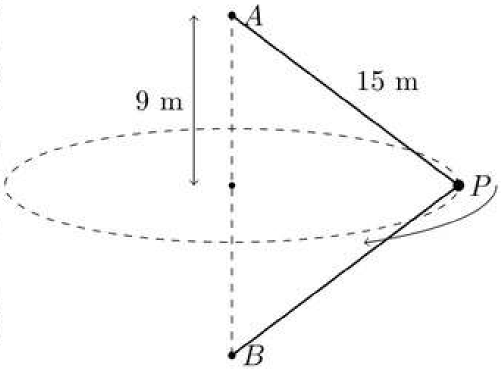
When freely suspended from A, the centre of mass must lie vertically below A. The horizontal distance from the central axis to A is the hemisphere's radius, $3r$. Constructing a right-angled triangle with the vertical axis:

$$\tan\theta = \frac{d_y}{3r} = \frac{\frac{79}{76}r}{3r} = \frac{79}{228}$$

$$\theta = \arctan\left(\frac{79}{228}\right) \approx \mathbf{19.1^\circ}$$

Question 2

The point B is vertically below the point A. One end of a light inextensible string is attached to A, and the other end is attached to B. A particle P of mass 7 kg is attached to the string so that the length AP is 15 m. The particle P moves at a constant angular speed of 4 radians per second in a horizontal circle with centre between A and B at a distance 9 m vertically below A (see diagram). The string has length 28 m, and both sections of the string are taut.
The tensions in the sections of the string AP and BP are T_A and T_B respectively. Find the value of T_A and the value of T_B. [5]



Solution

Let R be the radius of the horizontal circular path of P. By Pythagoras' theorem on $\triangle AOP$ (where O is the centre of the circle):

$$R = \sqrt{15^2 - 9^2} = \sqrt{225 - 81} = 12 \text{ m}$$

The total string length is 28 m, so the length of section BP is $28 - 15 = 13$ m. Let h be the vertical distance from P to B. Using Pythagoras on $\triangle BOP$:

$$h = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = 5 \text{ m}$$

Let α be the angle AP makes with the horizontal, and β be the angle BP makes with the horizontal.

$$\sin\alpha = \frac{9}{15} = \frac{3}{5}, \quad \cos\alpha = \frac{12}{15} = \frac{4}{5}$$

$$\sin\beta = \frac{5}{13}, \quad \cos\beta = \frac{12}{13}$$

Applying Newton's Second Law for particle P: **Vertically** (forces balance, acceleration is 0):

$$T_A\sin\alpha - T_B\sin\beta - mg = 0 \implies \frac{3}{5}T_A - \frac{5}{13}T_B = 7(10) = 70 \quad \text{— (1)}$$

Horizontally (centripetal force $F = m\omega^2 R$):

$$T_A \cos\alpha + T_B \cos\beta = m\omega^2 R \implies \frac{4}{5}T_A + \frac{12}{13}T_B = 7(4^2)(12) = 1344 \quad \text{— (2)}$$

To solve the system, we can multiply equation (1) by $\frac{12}{5}$ and add it to equation (2):

$$\left(\frac{36}{25}T_A - \frac{12}{13}T_B\right) + \left(\frac{4}{5}T_A + \frac{12}{13}T_B\right) = 70\left(\frac{12}{5}\right) + 1344$$

$$\frac{56}{25}T_A = 168 + 1344 = 1512 \implies T_A = 1512 \times \frac{25}{56} \implies T_A = \mathbf{675}\text{ N}$$

Substitute $T_A = 675$ back into equation (1):

$$\frac{3}{5}(675) - \frac{5}{13}T_B = 70 \implies 405 - 70 = \frac{5}{13}T_B \implies 335 = \frac{5}{13}T_B$$

$$T_B = 335 \times \frac{13}{5} \implies T_B = \mathbf{871}\text{ N}$$

Question 3

A particle P of mass 6.4 kg moves along a straight horizontal line. At time t s, the speed of P is v m s^{-1} and the distance of P from its starting position is x m. The forces acting on P are a constant driving force of magnitude 8 N and a resistive force of magnitude $0.5v^2$ N. Initially, P is at rest.

a. Find the speed of P when it has travelled 0.8 m. [5]

b. Find an expression for t in terms of v. [4]

Solution

a. Applying Newton's Second Law to the particle P gives:

$$F_{net} = ma \implies 8 - 0.5v^2 = 6.4a$$

Since we need velocity in terms of displacement, we use the identity $a = v\frac{dv}{dx}$:

$$6.4v\frac{dv}{dx} = 8 - 0.5v^2$$

Separate the variables to integrate:

$$\int \frac{6.4v}{8 - 0.5v^2}\,dv = \int dx$$

Let $u = 8 - 0.5v^2$, then $du = -v\,dv$. The integral becomes:

$$\int \frac{-6.4}{u}\,du = x + C \implies -6.4\ln|8 - 0.5v^2| = x + C$$

Using the initial conditions ($x = 0$ when $v = 0$):

$$-6.4\ln(8) = C$$

Substitute C back and solve for v when $x = 0.8$:

$$0.8 = -6.4\ln(8 - 0.5v^2) + 6.4\ln(8) \implies 0.8 = 6.4\ln\left(\frac{8}{8 - 0.5v^2}\right)$$

$$\ln\left(\frac{8}{8 - 0.5v^2}\right) = \frac{1}{8} \implies \frac{8}{8 - 0.5v^2} = e^{0.125}$$

$$8 - 0.5v^2 = 8e^{-0.125} \implies v^2 = 16(1 - e^{-0.125})$$

$$v = 4\sqrt{1 - e^{-0.125}} \approx \mathbf{1.37\ m\ s^{-1}}$$

b. To find t in terms of v, we use $a = \frac{dv}{dt}$:

$$6.4\frac{dv}{dt} = 8 - 0.5v^2 \implies \int dt = \int \frac{6.4}{8 - 0.5v^2}\, dv$$

Divide the numerator and denominator of the integral by 0.5:

$$t = \int \frac{12.8}{16 - v^2}\, dv$$

We decompose the fraction using partial fractions: $\frac{12.8}{(4-v)(4+v)} = \frac{A}{4-v} + \frac{B}{4+v}$.

$$12.8 = A(4 + v) + B(4 - v) \implies A = 1.6, \quad B = 1.6$$

Therefore, the integration becomes:

$$t = \int \left(\frac{1.6}{4 - v} + \frac{1.6}{4 + v}\right) dv = -1.6\ln|4 - v| + 1.6\ln|4 + v| + K$$

Given $t = 0$ when $v = 0$, we find $K = 0$. Using logarithmic properties:

$$t = \mathbf{1.6\ln\left(\frac{4 + v}{4 - v}\right)}$$

Question 4

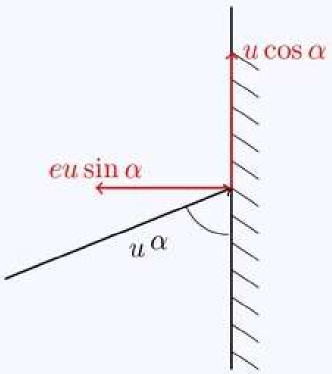
A particle P is moving on a fixed smooth horizontal surface with speed u. The particle collides with a fixed smooth vertical barrier. Immediately before the collision, the direction of motion of P makes an angle α with the barrier, where $\tan\alpha = \frac{12}{5}$. The kinetic energy of P after the collision is $\frac{4}{13}$ times its kinetic energy before the collision.

a. Find the coefficient of restitution between P and the barrier. [4]

b. A second smooth vertical barrier is fixed to the surface and to the first barrier. The angle between the two barriers is 120°. Particle P subsequently collides with the second barrier and rebounds immediately after with speed w at an angle θ with the second barrier. The coefficient of restitution is the same as in the first collision. Find the value of θ correct to 3 significant figures. [4]

Solution

a. The collision dynamics can be analysed by splitting the initial velocity u into components parallel and perpendicular to the wall.



From $\tan\alpha = \frac{12}{5}$, we obtain $\sin\alpha = \frac{12}{13}$ and $\cos\alpha = \frac{5}{13}$. The velocity component parallel to the barrier is unchanged: $v_{\parallel} = u\cos\alpha = \frac{5}{13}u$. The velocity component perpendicular to the barrier is scaled by the coefficient of restitution e: $v_{\perp} = eu\sin\alpha = \frac{12}{13}eu$.
The kinetic energy after the collision is related to the initial KE by:

$$\frac{1}{2}m(v_{\parallel}^2 + v_{\perp}^2) = \frac{4}{13}\left(\frac{1}{2}mu^2\right)$$

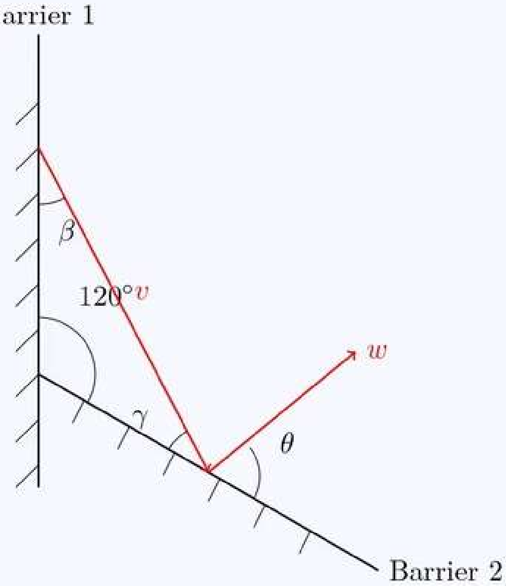
$$\left(\frac{5}{13}u\right)^2 + \left(\frac{12}{13}eu\right)^2 = \frac{4}{13}u^2$$

Dividing entirely by u^2:

$$\frac{25}{169} + \frac{144e^2}{169} = \frac{52}{169} \implies 144e^2 = 52 - 25 = 27$$

$$e^2 = \frac{27}{144} = \frac{3}{16} \implies e = \boldsymbol{\frac{\sqrt{3}}{4}}$$

b. To understand the subsequent collision, we model the geometric relationship between the two barriers and the particle's trajectory.



Let β be the angle of departure from the first barrier.

$$\tan\beta = \frac{v_\perp}{v_\parallel} = \frac{eu\sin\alpha}{u\cos\alpha} = e\tan\alpha = \left(\frac{\sqrt{3}}{4}\right)\left(\frac{12}{5}\right) = \frac{3\sqrt{3}}{5}$$

The particle travels towards the second barrier. Applying geometric properties of the triangle formed by the trajectory and the two intersecting barriers (120° apart), the angle of approach γ to the second barrier is:

$$\gamma = 180^\circ - 120^\circ - \beta = 60^\circ - \beta$$

Upon collision with the second barrier, the new departure angle θ follows the restitution rule $\tan\theta = e\tan\gamma$:

$$\tan\theta = e\tan(60^\circ - \beta) = \frac{\sqrt{3}}{4}\left[\frac{\tan 60^\circ - \tan\beta}{1 + \tan 60^\circ \tan\beta}\right]$$

Substitute $\tan 60^\circ = \sqrt{3}$ and $\tan\beta = \frac{3\sqrt{3}}{5}$:

$$\tan\theta = \frac{\sqrt{3}}{4}\left[\frac{\sqrt{3} - \frac{3\sqrt{3}}{5}}{1 + \sqrt{3}\left(\frac{3\sqrt{3}}{5}\right)}\right] = \frac{\sqrt{3}}{4}\left[\frac{\frac{2\sqrt{3}}{5}}{1 + \frac{9}{5}}\right] = \frac{\sqrt{3}}{4}\left[\frac{\frac{2\sqrt{3}}{5}}{\frac{14}{5}}\right] = \frac{\sqrt{3}}{4}\left(\frac{2\sqrt{3}}{14}\right) = \frac{3}{28}$$

$$\theta = \arctan\left(\frac{3}{28}\right) \approx \mathbf{6.12}^\circ$$

Question 5

A particle of mass m is attached to a fixed point O by a light inextensible string of length a. The particle initially hangs directly below O at the point A. The particle is then projected horizontally with speed $2\sqrt{ag}$, and begins to move in a vertical circle.

a. Find an expression for the tension in the string when it has turned through an angle θ. Give your answer in terms of m, g and θ. [4]

b. Find, in terms of a, the maximum height above the horizontal through A that is reached by the particle. [5]

Solution

a. Taking the lowest point A as the zero potential energy reference. After sweeping through an angle θ, the vertical height gained is $h = a - a\cos\theta = a(1-\cos\theta)$. Applying the Work-Energy principle:

$$\frac{1}{2}m(2\sqrt{ag})^2 - \frac{1}{2}mv^2 = mgh \implies 2mag - \frac{1}{2}mv^2 = mga(1-\cos\theta)$$

Rearranging to find the square of the instantaneous velocity v:

$$v^2 = 4ag - 2ag(1-\cos\theta) = 2ag(1+\cos\theta)$$

Resolving the forces along the radial direction towards O, we have the tension T and the radial component of weight:

$$T - mg\cos\theta = \frac{mv^2}{a}$$

Substitute v^2 into the equation:

$$T = mg\cos\theta + \frac{m}{a}[2ag(1+\cos\theta)] = mg\cos\theta + 2mg(1+\cos\theta)$$

$$T = \boldsymbol{mg(3\cos\theta + 2)}$$

b. The string becomes slack when the tension reaches zero ($T = 0$):

$$mg(3\cos\theta + 2) = 0 \implies \cos\theta = -\frac{2}{3}$$

Let this point be P_1. The height H_1 above A at this instant is:

$$H_1 = a(1-\cos\theta) = a\left(1 - \left(-\frac{2}{3}\right)\right) = \frac{5}{3}a$$

The velocity v_1 at this point squared is:

$$v_1^2 = 2ag(1+\cos\theta) = 2ag\left(1 - \frac{2}{3}\right) = \frac{2}{3}ag$$

Upon string slackening, the particle adopts projectile motion under gravity. The velocity vector is perpendicular to the radius. Let α be the angle the position vector makes with the upward vertical, so $\alpha = \pi - \theta$, giving $\cos\alpha = -\cos\theta = 2/3$. The velocity vector makes this same angle α with the horizontal. Therefore, the vertical component of the velocity v_y satisfies:

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(-\frac{2}{3}\right)^2 = \frac{5}{9}$$

$$v_y^2 = v_1^2\sin^2\theta = \left(\frac{2}{3}ag\right)\left(\frac{5}{9}\right) = \frac{10}{27}ag$$

The further vertical height H_2 attained during projectile flight until the vertical velocity becomes zero is:

$$H_2 = \frac{v_y^2}{2g} = \frac{\frac{10}{27}ag}{2g} = \frac{5}{27}a$$

The total maximum height above A is:

$$H_{max} = H_1 + H_2 = \frac{5}{3}a + \frac{5}{27}a = \frac{45}{27}a + \frac{5}{27}a = \boldsymbol{\frac{50}{27}a}$$

Question 6

A light elastic string has natural length 2.5 m and modulus of elasticity 15 N. One end of the string is attached to the fixed point O, and the other end is attached to a particle P of mass 1.5 kg. The particle P is held at O and then released from rest so that it begins to fall vertically. In its subsequent motion, P moves with velocity v m s^{-1} when the extension of the string is x m. The maximum extension of the string during this motion is E m.

a. Show that, for $0 < x < E$, $v^2 = -4(x^2 - 5x - 12.5)$. [3]

b. Find the value of E. [2]

c. Find the maximum speed of P. [1]

Solution

a. Set the initial gravitational potential energy at O to 0. When the extension is x, the particle has fallen a total distance of $L_0 + x = 2.5 + x$. By the conservation of mechanical energy:

$$\text{Loss in GPE} = \text{Gain in KE} + \text{Gain in EPE}$$

$$mg(2.5 + x) = \frac{1}{2}mv^2 + \frac{\lambda x^2}{2L_0}$$

Given $m = 1.5$, $g = 10$, $\lambda = 15$, and $L_0 = 2.5$:

$$1.5(10)(2.5 + x) = \frac{1}{2}(1.5)v^2 + \frac{15x^2}{2(2.5)} \implies 15(2.5 + x) = 0.75v^2 + 3x^2$$

Divide the entire equation by 0.75:

$$20(2.5 + x) = v^2 + 4x^2 \implies 50 + 20x = v^2 + 4x^2$$

Rearranging for v^2:

$$v^2 = \mathbf{-4(x^2 - 5x - 12.5)}$$

b. The maximum extension E occurs when the velocity instantaneously drops to 0. Set $v = 0$:

$$-4(E^2 - 5E - 12.5) = 0 \implies E^2 - 5E - 12.5 = 0$$

Using the quadratic formula:

$$E = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(-12.5)}}{2} = \frac{5 \pm \sqrt{25 + 50}}{2} = \frac{5 \pm \sqrt{75}}{2} = \frac{5 \pm 5\sqrt{3}}{2}$$

Since the extension must be positive during the fall downward, we discard the negative root:

$$E = \frac{5 + 5\sqrt{3}}{2} \approx \mathbf{6.83}\text{ m}$$

c. To find the maximum speed, we maximize the quadratic expression for v^2. Completing the square for v^2:

$$v^2 = -4\left[(x - 2.5)^2 - 2.5^2 - 12.5\right] = -4\left[(x - 2.5)^2 - 18.75\right]$$

$$v^2 = 75 - 4(x - 2.5)^2$$

The maximum value of v^2 is reached when the squared term is zero, which happens at $x = 2.5$.

$$v_{max}^2 = 75 \implies v_{max} = \sqrt{75} = 5\sqrt{3} \approx \mathbf{8.66}\text{ m s}^{-1}$$

Question 7

The points O and A lie on a horizontal plane 10 m apart. The point A lies at the foot of another plane which is inclined at an angle of α above the horizontal such that $\tan\alpha = \frac{4}{3}$. The point B is on the inclined plane at a distance of 5 m from A measured along a line of greatest slope. The points O, A and B are in the same vertical plane and angle OAB is obtuse.

A particle is projected from O at an angle θ to the horizontal with initial speed u m s^{-1} towards the inclined plane. The particle moves freely under gravity. The particle strikes the inclined plane at the point B when its direction of motion is perpendicular to the inclined plane. Find, in either order, the value of u and the value of θ. [6]

Solution

Establish a Cartesian coordinate system with the origin at O. The horizontal displacement of B is the distance from O to A plus the horizontal component of AB:

$$x_B = 10 + 5\cos\alpha$$

Using $\tan\alpha = \frac{4}{3}$, we have a 3-4-5 triangle, so $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = \frac{4}{5}$.

$$x_B = 10 + 5\left(\frac{3}{5}\right) = 13 \text{ m}$$

The vertical displacement of B is:

$$y_B = 5\sin\alpha = 5\left(\frac{4}{5}\right) = 4 \text{ m}$$

The particle hits B perpendicularly to the inclined plane. The incline has a gradient of $\tan\alpha = \frac{4}{3}$. The direction perpendicular to it has a gradient of $-\frac{3}{4}$. Thus, the velocity vector at B satisfies $\frac{v_y}{v_x} = -\frac{3}{4}$. The kinematic equations for the projectile at time t are:

$$v_x = u\cos\theta, \quad v_y = u\sin\theta - gt, \quad x = (u\cos\theta)t$$

At point B ($x = 13$), time $t = \frac{13}{u\cos\theta}$. Substituting this into the velocity gradient:

$$\frac{u\sin\theta - 10\left(\frac{13}{u\cos\theta}\right)}{u\cos\theta} = -\frac{3}{4} \implies \tan\theta - \frac{130}{u^2\cos^2\theta} = -0.75 \implies \frac{130}{u^2\cos^2\theta} = \tan\theta + 0.75 \quad \text{— (1)}$$

Additionally, the trajectory equation must be satisfied at B $(13, 4)$:

$$y_B = x_B\tan\theta - \frac{gx_B^2}{2u^2\cos^2\theta} \implies 4 = 13\tan\theta - \frac{10(13)^2}{2u^2\cos^2\theta}$$

$$4 = 13\tan\theta - \frac{845}{u^2\cos^2\theta} \quad \text{— (2)}$$

Multiply equation (1) by $\frac{845}{130} = 6.5$ to isolate the u term:

$$\frac{845}{u^2\cos^2\theta} = 6.5\tan\theta + 4.875$$

Substitute this into equation (2):

$$4 = 13\tan\theta - (6.5\tan\theta + 4.875) \implies 4 = 6.5\tan\theta - 4.875$$

$$6.5\tan\theta = 8.875 \implies \tan\theta = \frac{8.875}{6.5} = \frac{71}{52}$$

$$\theta = \arctan\left(\frac{71}{52}\right) \approx \mathbf{53.8}^\circ$$

Now substitute $\tan\theta = \frac{71}{52}$ back into equation (1) to solve for u:

$$\frac{130}{u^2\cos^2\theta} = \frac{71}{52} + \frac{3}{4} = \frac{71}{52} + \frac{39}{52} = \frac{110}{52} = \frac{55}{26}$$

Rearranging for u^2:

$$u^2 = \frac{130 \times 26}{55\cos^2\theta} = \frac{676}{11}\sec^2\theta = \frac{676}{11}(1 + \tan^2\theta)$$

$$u^2 = \frac{676}{11}\left(1 + \left(\frac{71}{52}\right)^2\right) = \frac{676}{11}\left(1 + \frac{5041}{2704}\right) = \frac{676}{11}\left(\frac{7745}{2704}\right)$$

Noting that $2704 = 4 \times 676$, this simplifies cleanly:

$$u^2 = \frac{7745}{44} \approx 176.02$$

$$u = \sqrt{\frac{7745}{44}} \approx \mathbf{13.3}\ \mathrm{m\ s}^{-\mathbf{1}}$$