A-Level Mathematics Examination Review

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Questions and Solutions

Question 1
Solve the equation

$$3x^2 - 8 - \frac{16}{x^2} = 0.$$

[3]

Solution

To clear the denominator, we multiply the entire equation by x^2 (assuming $x \neq 0$):

$$3x^4 - 8x^2 - 16 = 0$$

This is a quadratic in terms of x^2. Let $u = x^2$. Substituting this into the equation gives:

$$3u^2 - 8u - 16 = 0$$

We can factorize this quadratic by finding two numbers that multiply to $3 \times (-16) = -48$ and add to -8. These numbers are -12 and 4.

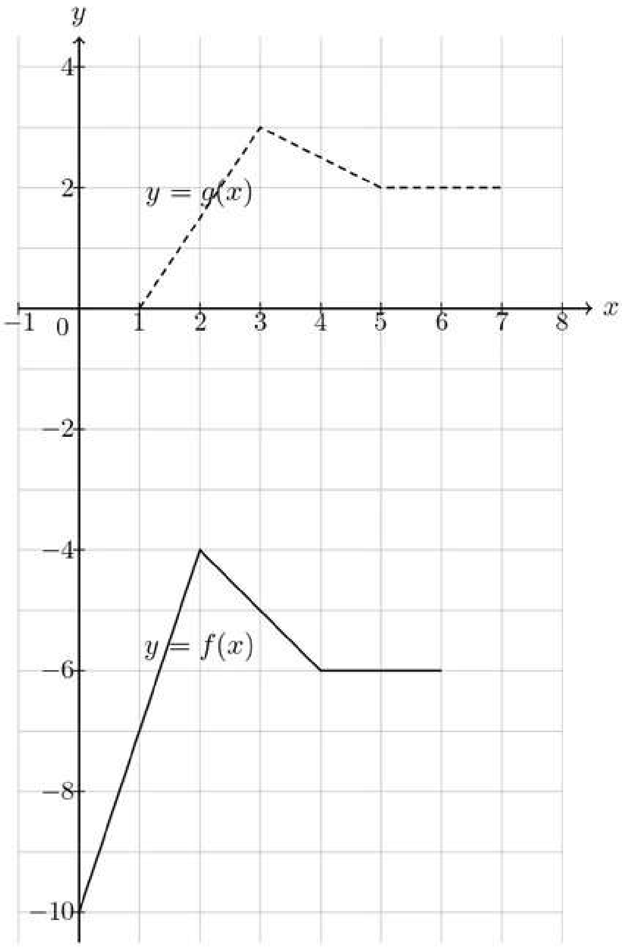
$$3u^2 - 12u + 4u - 16 = 0$$

$$3u(u - 4) + 4(u - 4) = 0$$

$$(3u + 4)(u - 4) = 0$$

This gives $u = -\frac{4}{3}$ or $u = 4$. Since $u = x^2$ and x must be a real number, $x^2 \geq 0$. We reject $u = -\frac{4}{3}$. Thus, $x^2 = 4$, which yields: $x = \pm 2$

Question 2
In the diagram, the graph with equation $y = f(x)$ is shown with solid lines and the graph with equation $y = g(x)$ is shown with broken lines.



(a) Describe fully a sequence of transformations which transforms the graph of $y = f(x)$ to the graph of $y = g(x)$. You should make clear the order in which the transformations are applied. [5]

Solution

Let's analyze the key vertices of both graphs. The vertices of $y = f(x)$ are $(0, -10)$, $(2, -4)$, and $(4, -6)$. The corresponding vertices of $y = g(x)$ are $(1, 0)$, $(3, 3)$, and $(5, 2)$.
Comparing the x-coordinates: $0 \to 1$, $2 \to 3$, $4 \to 5$. This is a consistent shift of $+1$ in the x-direction. Comparing the y-coordinates: $-10 \to 0$, $-4 \to 3$, $-6 \to 2$. We seek a linear relationship $y' = ay + c$. Using the first two points: $-10a + c = 0$ $-4a + c = 3$ Subtracting the first from the second gives $6a = 3 \Rightarrow a = \frac{1}{2}$. Substituting a back yields $c = 5$. So, the transformation is $y' = \frac{1}{2}y + 5$.
The sequence of transformations (order matters) can be stated as: **1. A vertical stretch parallel to the y-axis with a scale factor of $\frac{1}{2}$.**
2. A translation by the vector $\binom{1}{5}$.

(b) Find an expression for $g(x)$ in the form $af(x + b) + c$, where a, b and c are constants. [2]

Solution

Based on the transformations identified in part (a), a translation of $+1$ in the x-direction corresponds to $f(x - 1)$. The vertical stretch by $\frac{1}{2}$ and translation of $+5$ applies to the entire function, yielding $\frac{1}{2}f(x - 1) + 5$. Comparing this to $af(x + b) + c$, we have $a = \frac{1}{2}$, $b = -1$, and $c = 5$. $g(x) = \frac{1}{2}f(x - 1) + 5$

Question 3
The perpendicular bisector of the line joining $(4, 3)$ and $(-2, 6)$ meets the line $x - 4y - 4 = 0$ at the point P. Find the coordinates of P. [6]

Solution

First, find the midpoint M of the segment connecting $(4, 3)$ and $(-2, 6)$:

$$M = \left(\frac{4 + (-2)}{2}, \frac{3 + 6}{2}\right) = \left(1, \frac{9}{2}\right)$$

Next, determine the gradient m_1 of the line segment:

$$m_1 = \frac{6 - 3}{-2 - 4} = \frac{3}{-6} = -\frac{1}{2}$$

The gradient m_2 of the perpendicular bisector is the negative reciprocal of m_1:

$$m_2 = -\frac{1}{m_1} = 2$$

Using the point-slope form, the equation of the perpendicular bisector passing through M is:

$$y - \frac{9}{2} = 2(x - 1)$$

$$2y - 9 = 4x - 4 \implies 4x - 2y + 5 = 0$$

We need to find the intersection of this bisector with the line $x - 4y - 4 = 0 \implies x = 4y + 4$. Substitute x into the bisector's equation:

$$4(4y + 4) - 2y + 5 = 0$$

$$16y + 16 - 2y + 5 = 0 \implies 14y + 21 = 0 \implies y = -\frac{3}{2}$$

Substitute $y = -\frac{3}{2}$ back into $x = 4y + 4$:

$$x = 4\left(-\frac{3}{2}\right) + 4 = -6 + 4 = -2$$

Thus, the coordinates of P are: $P(-2, -1.5)$

Question 4
(a) (i) Find the first three terms, in ascending powers of x, in the expansion of $(3 - \frac{1}{2}x)^4$. [2]

Solution

Using the binomial theorem:

$$(3 - \frac{1}{2}x)^4 = \binom{4}{0}3^4 + \binom{4}{1}3^3\left(-\frac{1}{2}x\right)^1 + \binom{4}{2}3^2\left(-\frac{1}{2}x\right)^2 + \dots$$

$$= 1(81) + 4(27)\left(-\frac{1}{2}x\right) + 6(9)\left(\frac{1}{4}x^2\right)$$

$= 81 - 54x + \frac{27}{2}x^2$

(ii) Find the first three terms, in ascending powers of x, in the expansion of $(2+ax)^3$. Give your answer in terms of the constant a. [2]

Solution

Similarly, applying the binomial theorem:

$$(2+ax)^3 = \binom{3}{0}2^3 + \binom{3}{1}2^2(ax)^1 + \binom{3}{2}2^1(ax)^2 + \dots$$

$$= 1(8) + 3(4)(ax) + 3(2)(a^2x^2)$$

$= 8 + 12ax + 6a^2x^2$

(b) It is now given that the coefficient of x^2 in the expansion of $(3-\frac{1}{2}x)^4(2+ax)^3$ is -108. Find the value of a. [4]

Solution

We multiply the expansions from part (a):

$$\left(81 - 54x + \frac{27}{2}x^2\right)(8 + 12ax + 6a^2x^2)$$

We only care about the terms that produce x^2:

$$(81)(6a^2x^2) + (-54x)(12ax) + \left(\frac{27}{2}x^2\right)(8)$$

$$= 486a^2x^2 - 648ax^2 + 108x^2$$

The coefficient of x^2 is $486a^2 - 648a + 108$. We are given that this is equal to -108:

$$486a^2 - 648a + 108 = -108$$

$$486a^2 - 648a + 216 = 0$$

Dividing the entire equation by 54:

$$9a^2 - 12a + 4 = 0$$

This is a perfect square trinomial:

$$(3a-2)^2 = 0 \implies 3a - 2 = 0$$

$a = \frac{2}{3}$

Question 5

The equation of a curve is such that $\frac{d^2y}{dx^2} = 6x^2 + p$ where p is a constant. The curve has minimum points at $(2, 3)$ and $(-2, 3)$. Find the coordinates of the other stationary point of the curve. [7]

Solution

Since $(2, 3)$ and $(-2, 3)$ are minimum points, they are stationary points, meaning $\frac{dy}{dx} = 0$ at $x = 2$ and $x = -2$. First, let's integrate $\frac{d^2y}{dx^2}$ to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \int (6x^2 + p)\, dx = 2x^3 + px + C$$

Applying the condition $\frac{dy}{dx} = 0$ at $x = 2$:

$$2(2)^3 + p(2) + C = 0 \implies 16 + 2p + C = 0 \quad \text{(Equation 1)}$$

Applying the condition $\frac{dy}{dx} = 0$ at $x = -2$:

$$2(-2)^3 + p(-2) + C = 0 \implies -16 - 2p + C = 0 \quad \text{(Equation 2)}$$

Adding Equation 1 and Equation 2 yields $2C = 0 \implies C = 0$. Substitute $C = 0$ into Equation 1: $16 + 2p = 0 \implies p = -8$. So, $\frac{dy}{dx} = 2x^3 - 8x$.
To find the other stationary point, set $\frac{dy}{dx} = 0$:

$$2x^3 - 8x = 0 \implies 2x(x^2 - 4) = 0 \implies x = 0, \text{ or } x = \pm 2.$$

The other stationary point occurs at $x = 0$. Now, integrate to find y:

$$y = \int (2x^3 - 8x)\, dx = \frac{1}{2}x^4 - 4x^2 + K$$

We know the curve passes through $(2, 3)$. Substitute $x = 2$ and $y = 3$:

$$3 = \frac{1}{2}(16) - 4(4) + K \implies 3 = 8 - 16 + K \implies K = 11$$

The equation of the curve is $y = \frac{1}{2}x^4 - 4x^2 + 11$. At $x = 0$, $y = 11$. **The coordinates of the other stationary point are $(0, 11)$.**

Question 6
Solve the equation $cos^{-1}(sin^2x + \frac{3}{2}sin\ x) = \frac{2}{3}\pi$ for $0 \leq x \leq 2\pi$. [4]

Solution

Taking the cosine of both sides gives:

$$\sin^2 x + \frac{3}{2}\sin x = \cos\left(\frac{2}{3}\pi\right)$$

Since $\cos\left(\frac{2}{3}\pi\right) = -\frac{1}{2}$, we have:

$$\sin^2 x + \frac{3}{2}\sin x = -\frac{1}{2}$$

Multiply by 2 to clear the fraction:

$$2\sin^2 x + 3\sin x + 1 = 0$$

Let $S = \sin x$. The quadratic is $2S^2 + 3S + 1 = 0$, which factors as:

$$(2S + 1)(S + 1) = 0$$

So, $\sin x = -\frac{1}{2}$ or $\sin x = -1$. For $\sin x = -\frac{1}{2}$ in the interval $0 \le x \le 2\pi$: $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ and $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$. For $\sin x = -1$ in the interval $0 \le x \le 2\pi$: $x = \frac{3\pi}{2}$. $x = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Question 7

(a) The second, fourth and last terms of an arithmetic progression are 60, 52 and -28 respectively. Find the sum of all of the terms in the progression. [4]

Solution

Let a be the first term and d be the common difference. $u_2 = a + d = 60$ $u_4 = a + 3d = 52$ Subtracting the first equation from the second gives: $2d = -8 \implies d = -4$. Substitute d back to find a: $a - 4 = 60 \implies a = 64$. The last term is given by $u_n = a + (n-1)d = -28$. $64 + (n-1)(-4) = -28 \implies -4(n-1) = -92 \implies n - 1 = 23 \implies n = 24$. The sum of an arithmetic progression is $S_n = \frac{n}{2}(a + l)$, where l is the last term.

$$S_{24} = \frac{24}{2}(64 + (-28)) = 12(36) = 432$$

$S = 432$

(b) The second, third and fourth terms of a geometric progression are 9, $k+4$ and k^2 respectively, where k is a negative constant.

(i) Find the value of k. [3]

Solution

In a geometric progression, the common ratio r is constant:

$$r = \frac{u_3}{u_2} = \frac{u_4}{u_3} \implies \frac{k+4}{9} = \frac{k^2}{k+4}$$

Cross-multiplying yields:

$$(k+4)^2 = 9k^2 \implies k^2 + 8k + 16 = 9k^2$$

$$8k^2 - 8k - 16 = 0 \implies k^2 - k - 2 = 0$$

Factoring this quadratic:

$$(k-2)(k+1) = 0 \implies k = 2 \text{ or } k = -1$$

Since we are given that k is a negative constant, we reject $k = 2$. $k = -1$

(ii) Find the sum to infinity of the progression. [3]

Solution

Using $k = -1$, the common ratio is $r = \frac{-1+4}{9} = \frac{3}{9} = \frac{1}{3}$. The second term is $ar = 9 \implies a\left(\frac{1}{3}\right) = 9 \implies a = 27$. The sum to infinity is given by $S_\infty = \frac{a}{1-r}$:

$$S_\infty = \frac{27}{1 - \frac{1}{3}} = \frac{27}{\frac{2}{3}} = \frac{81}{2}$$

$S_\infty = \frac{81}{2}$

Question 8

A diameter of a circle has end-points at $(-2, 4)$ and $(8, 8)$.

(a) Find an equation of the circle. [3]

Solution

The center of the circle C is the midpoint of the diameter:

$$C = \left(\frac{-2+8}{2}, \frac{4+8}{2}\right) = (3, 6)$$

The radius squared (r^2) is the squared distance from the center to one of the endpoints, say $(8, 8)$:

$$r^2 = (8-3)^2 + (8-6)^2 = 5^2 + 2^2 = 25 + 4 = 29$$

The equation of the circle is $(x-a)^2 + (y-b)^2 = r^2$: $(x-3)^2 + (y-6)^2 = 29$

(b) The line with equation $y = x + 6$ intersects the circle at the points P and Q. Find the exact distance PQ. Give your answer in the form $a\sqrt{b}$, where a and b are both integers greater than 1. [4]

Solution

Substitute the line equation $y = x + 6$ into the circle's equation to find the points of intersection:

$$(x-3)^2 + (x+6-6)^2 = 29$$
$$(x-3)^2 + x^2 = 29$$
$$x^2 - 6x + 9 + x^2 = 29 \implies 2x^2 - 6x - 20 = 0$$

Divide by 2 to simplify:

$$x^2 - 3x - 10 = 0 \implies (x-5)(x+2) = 0$$

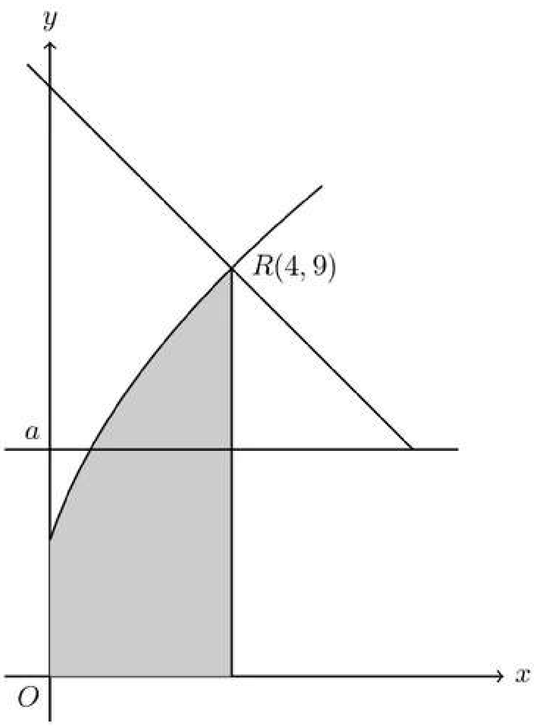
This yields $x = 5$ or $x = -2$. Substitute these x values back into $y = x + 6$: When $x = 5$, $y = 11 \implies P(5, 11)$ When $x = -2$, $y = 4 \implies Q(-2, 4)$ The distance PQ is:

$$PQ = \sqrt{(-2-5)^2 + (4-11)^2} = \sqrt{(-7)^2 + (-7)^2} = \sqrt{49+49} = \sqrt{98}$$

Simplifying the surd: $\sqrt{98} = \sqrt{49 \times 2} = 7\sqrt{2}$. Here $a = 7$ and $b = 2$, both integers greater than 1. $PQ = 7\sqrt{2}$

Question 9

The diagram shows the curve with equation $y = 3\sqrt{2x+1}$ the normal to the curve at the point $R(4, 9)$ and the line $y = a$, where a is a positive constant.



(a) Find the area of the shaded region. [4]

Solution

The shaded area is bounded by the curve, the x-axis, the y-axis ($x = 0$), and the line $x = 4$.

$$\text{Area} = \int_0^4 3\sqrt{2x+1}\,dx = \int_0^4 3(2x+1)^{1/2}\,dx$$

We can use substitution or direct integration. Let $u = 2x + 1$, then $du = 2dx$, or $dx = \frac{du}{2}$. The limits change from $x = 0 \to u = 1$, and $x = 4 \to u = 9$.

$$\text{Area} = \int_1^9 3u^{1/2}\frac{du}{2} = \frac{3}{2}\int_1^9 u^{1/2}\,du = \frac{3}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^9 = \left[u^{3/2}\right]_1^9$$

$$= 9^{3/2} - 1^{3/2} = (\sqrt{9})^3 - 1 = 3^3 - 1 = 27 - 1 = 26$$

Area = 26

(b) Find the exact value of a for which the triangular area enclosed by the y-axis, the normal to the curve at R and the line $y = a$ is equal to the area found in 9(a). [7]

Solution

First, we find the equation of the normal to the curve at $R(4, 9)$. The derivative is $\frac{dy}{dx} = 3\cdot\frac{1}{2}(2x+1)^{-1/2}\cdot 2 = \frac{3}{\sqrt{2x+1}}$. At $x = 4$, the gradient of the tangent is $m_t = \frac{3}{\sqrt{2(4)+1}} = \frac{3}{3} = 1$. The gradient of the normal is $m_n = -\frac{1}{m_t} = -1$. Using point-slope form, the equation of the normal line at $(4, 9)$ is:

$$y - 9 = -1(x-4) \implies y = -x + 13$$

The triangle in question is bounded by the y-axis ($x = 0$), the normal line ($y = -x+13$), and the horizontal line $y = a$. The normal line intersects the y-axis at $(0, 13)$. The horizontal line $y = a$ intersects the y-axis at $(0, a)$. The intersection of the normal line and $y = a$ is: $a = -x + 13 \implies x = 13 - a$. The vertices of the triangle are $(0, 13)$, $(0, a)$, and $(13 - a, a)$. This forms a right-angled triangle with a vertical height of $|13 - a|$ and a horizontal base of $|13 - a|$.

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(13-a)^2$$

We are given that this area equals the area from part (a), which is 26:

$$\frac{1}{2}(13-a)^2 = 26 \implies (13-a)^2 = 52 \implies 13 - a = \pm\sqrt{52} = \pm 2\sqrt{13}$$

$a = 13 \pm 2\sqrt{13}$. Based on the diagram, the intersection of $y = a$ and the normal lies in the first quadrant, meaning $x = 13 - a > 0 \implies a < 13$. Therefore, we reject the positive root. $a = 13 - 2\sqrt{13}$

Question 10

A function f is defined by $f(x) = px^2 - 2x + p + 4$ for $x \in \mathbb{R}$, where p is a constant.

(a) Find the exact values of p for which the equation $f(x) = 0$ has repeated roots. [3]

Solution

For a quadratic equation $ax^2 + bx + c = 0$ to have repeated roots, the discriminant $\Delta = b^2 - 4ac$ must equal 0. Here, $a = p$, $b = -2$, and $c = p + 4$.

$$\Delta = (-2)^2 - 4(p)(p+4) = 0$$

$$4 - 4p^2 - 16p = 0$$

Dividing by -4 yields:

$$p^2 + 4p - 1 = 0$$

Using the quadratic formula to solve for p:

$$p = \frac{-4 \pm \sqrt{16 - 4(1)(-1)}}{2} = \frac{-4 \pm \sqrt{20}}{2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

$p = -2 \pm \sqrt{5}$

A function g is defined by $g(x) = px^2 - 2x + p + 4$ for $x \geq a$ where p and a are positive constants.
(b) (i) Show that the least value of a for which the function g^{-1} exists is $\frac{1}{p}$. [2]

Solution

For the inverse function g^{-1} to exist, $g(x)$ must be a one-to-one function. Since $g(x)$ is a quadratic function opening upwards (p is a positive constant), it represents a parabola. To be one-to-one, its domain must be restricted to one side of the vertex. The x-coordinate of the vertex of $y = px^2 - 2x + p + 4$ is found where $g'(x) = 0$:

$$g'(x) = 2px - 2 = 0 \implies 2px = 2 \implies x = \frac{1}{p}$$

For $x \geq a$, the function will be strictly increasing (and thus one-to-one) if a starts at or after the vertex. Therefore, the least possible value for the domain restriction a is the x-coordinate of the vertex. **Least value of $a = \frac{1}{p}$ (Shown)**

(ii) Given that $p = 2$ and g^{-1} does exist, find an expression for $g^{-1}(x)$. [4]

Solution

Substitute $p = 2$ into the function: $g(x) = 2x^2 - 2x + 6$. From part (b)(i), we know the domain is restricted to $x \geq \frac{1}{2}$. To find the inverse, let $y = 2x^2 - 2x + 6$ and complete the square for x:

$$y = 2(x^2 - x) + 6 = 2\left(x - \frac{1}{2}\right)^2 - 2\left(\frac{1}{4}\right) + 6$$

$$y = 2\left(x - \frac{1}{2}\right)^2 - \frac{1}{2} + \frac{12}{2} = 2\left(x - \frac{1}{2}\right)^2 + \frac{11}{2}$$

Now, isolate x:

$$y - \frac{11}{2} = 2\left(x - \frac{1}{2}\right)^2 \implies \frac{2y - 11}{2} = 2\left(x - \frac{1}{2}\right)^2$$

$$\left(x - \frac{1}{2}\right)^2 = \frac{2y - 11}{4}$$

$$x - \frac{1}{2} = \pm\frac{\sqrt{2y - 11}}{2}$$

Since the domain is restricted to $x \geq \frac{1}{2}$, we take the positive root to ensure $x - \frac{1}{2} \geq 0$:

$$x = \frac{1}{2} + \frac{\sqrt{2y - 11}}{2} = \frac{1 + \sqrt{2y - 11}}{2}$$

Replace y with x to express the inverse function: $g^{-1}(x) = \frac{1+\sqrt{2x-11}}{2}$