1 The equation of a curve is such that $dy/dx = 2x - 6x^{\frac{1}{2}}$. The curve passes through the point $(4, -9)$. [4]

Find the equation of the curve.

$y = \int (2x - 6x^{1/2})\,dx \longrightarrow y = \frac{2x^2}{2} - \frac{6x^{3/2}}{3/2} + c$

$y = x^2 - 4x^{3/2} + c$

$-9 = (4)^2 - 4(4)^{3/2} + c \longrightarrow c = 7$

Equation of the curve: $y = x^2 - 4x^{3/2} + 7$

2

(a) Describe fully a sequence of two transformations which transforms the graph of $y = f(x)$ to the graph of $y = f(4 - x)$. [3]

Sequence 1	Sequence 2
1. Reflection in the y-axis 2. Translation by the vector $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$	1. Translation by the vector $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ 2. Reflection in the y-axis

(b) The curve with equation $y = x^3 - 3x - 4$ is stretched with scale factor $\frac{1}{2}$ in the x-direction and then translated by $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

Find and simplify the equation of the transformed curve. [2]

$y = (2x)^3 - 3(2x) - 4$

Stretch: $y = 8x^3 - 6x - 4$

Translation: $y = 8x^3 - 6x - 4 - 3$

$y = 8x^3 - 6x - 7$

3 The equation of a curve is $y = kx^2 - 5x - 6$, and the equation of a line is $y = 3x - 7k$. [4]

Find the set of values of the constant k for which the line intersects the curve.

$kx^2 - 5x - 6 = 3x - 7k$

$kx^2 - 8x + (7k - 6) = 0$

$b^2 - 4ac \geq 0 \longrightarrow (-8)^2 - 4(k)(7k-6) \geq 0$

$64 - 28k^2 + 24k \geq 0$

$7k^2 - 6k - 16 \leq 0$

$7k^2 - 8k + 2k - 16 \leq 0$

$(7k + 8)(k - 2) \leq 0$

$-\frac{8}{7} \leq k \leq 2$

4 The coefficient of x^2 in the expansion of $(2-qx)^4 - \left(1+\frac{8}{q}x\right)^6$ is 324.

Find the possible values of the constant q. [5]

$\binom{4}{2}(2)^{4-2}\cdot(-qx)^2 - \binom{6}{2}(1)^{6-2}\cdot\left(\frac{8}{q}x\right)^2 = 324x^2$

$24q^2x^2 - \frac{960}{q^2}x^2 = 324x^2$

$\left(24q^2 - \frac{960}{q^2} = 324\right) \times q^2$

$24q^4 - 960 = 324q^2$

$24q^4 - 324q^2 - 960 = 0$

Let $u = q^2$

$24u^2 - 324u - 960 = 0$

$2u^2 - 27u - 80 = 0$

$2u^2 - 32u + 5u - 80 = 0 \rightarrow 2u(u-16) + 5(u-16) = 0$

$(2u+5)(u-16) = 0$

$u = -\frac{5}{2}$ or $u = 16$

$q^2 = 16$

$q^2 = -\frac{5}{2}$ $q = \pm 4$

no real solution

5 **(a)** Prove the identity $\frac{1-\sin\theta}{\cos\theta}+\frac{\cos\theta}{1-\sin\theta}\equiv\frac{2}{\cos\theta}$. [3]

$$\frac{(1-\sin\theta)(1-\sin\theta)+(\cos\theta)(\cos\theta)}{(\cos\theta)(1-\sin\theta)}$$

$$\frac{(1-\sin\theta)^2+\cos^2\theta}{\cos\theta(1-\sin\theta)}$$

$$\frac{(1-\sin\theta)^2+1^2-\sin^2\theta}{\cos\theta(1-\sin\theta)}$$

$$\frac{(1-\sin\theta)^2+(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1-\sin\theta)}$$

$$\frac{\cancel{(1-\sin\theta)}(1-\sin\theta+1+\sin\theta)}{\cos\theta\cancel{(1-\sin\theta)}}$$

$$=\frac{2}{\cos\theta}$$

(b) Hence, solve the equation $\dfrac{1-\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{1-\sin\theta}=\dfrac{\tan^3\theta}{\sin\theta}$ for $0° \le \theta \le 360°$. [4]

$$\frac{2}{\cos\theta} = \frac{\tan^3\theta}{\sin\theta}$$

$$2 = \tan^3\theta \cdot \frac{\cos\theta}{\sin\theta}$$

$$2 = \tan^3\theta / \tan\theta \longrightarrow \tan^2\theta = 2$$

$$\tan\theta = \pm\sqrt{2}$$

$$\alpha = \tan^{-1}(\sqrt{2}) \approx 54.7°$$

$$\theta = 54.7°,\ 180-54.7°,\ 180+54.7°,\ 360-54.7°$$

$$= 54.7°,\ 125.3°,\ 234.7°,\ 305.3°$$

6 The coordinates of three points, P, Q and R, are $(0, p)$, $(8, 6)$ and $(r, 10)$ respectively, where p and r are constants. It is given that PQ is perpendicular to QR.

(a) Show that $p = 2r - 10$. [2]

$$m_{PQ} = \frac{6-p}{8-0} = \frac{6-p}{8}$$

$$m_{QR} = \frac{10-6}{r-8} = \frac{4}{r-8}$$

$$m_{PQ} \times m_{QR} = -1$$

$$m_{PQ} = -1/m_{QR}$$

$$\frac{6-p}{8_2} = \frac{-(r-8)}{4_1} \rightarrow 6-p = 16-2r$$

$$6-16+2r = p$$

$$p = 2r - 10$$

It is further given that the length of PR is $\sqrt{85}$.

(b) Find the possible values of p and r. [5]

$|PR| = \sqrt{85} = \sqrt{(r-0)^2 + (10-p)^2}$

$r^2 + (10-p)^2 = 85$

$r^2 + (10-(2r-10))^2 = 85$

$r^2 + (20-2r)^2 = 85$

$r^2 + 4r^2 - 80r + 400 - 85 = 0$

$5r^2 - 80r + 315 = 0$

$r^2 - 16r + 63 = 0$

$r^2 - 9r - 6r + 63 = 0$

$r(r-9) - 6(r-9) = 0 \rightarrow (r-6)(r-9) = 0$

$r = 9$ or $r = 7$

· If $r = 9$, $p = 2(9) - 10 = 8$

If $r = 7$, $p = 2(7) - 10 = 4$

$(r = 9, p = 8)$ or $(r = 7, p = 4)$

7 A curve has equation

$$y = 3x^{-1/2} - 2x^{-3/2}.$$

The curve has a single stationary point when $x = k$, where $k > 0$.

(a) Find the value of k. [4]

$$\frac{dy}{dx} = 3\left(\frac{-1}{2}\right)x^{-1/2-1} - 2\left(\frac{-3}{2}\right)x^{-3/2-1}$$

$$\frac{dy}{dx} = -\frac{3}{2}x^{-3/2} + 3x^{-5/2} \longrightarrow 0 = -\frac{3}{2}x^{-3/2} + 3x^{-5/2}$$

$$\frac{-3}{2x^{3/2}} + \frac{3}{x^{5/2}} = 0$$

$$\frac{\cancel{3}1}{x^{5/2}} = \frac{\cancel{3}1}{2x^{3/2}} \qquad 2 = x^{5/2}/x^{3/2}$$

$$x^{5/2-3/2} = 2$$

$$x = 2$$

$$k = 2$$

(b) Find d^2y/dx^2, and hence determine whether the stationary point is a maximum or a minimum. [2]

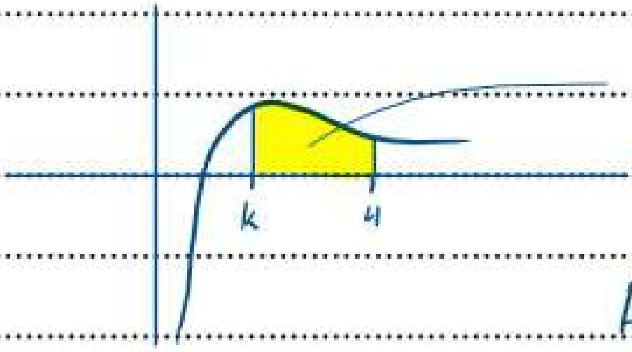
$$\frac{d^2y}{dx^2} = -\frac{3}{2}\left(-\frac{3}{2}\right)x^{-5/2} + 3\left(\frac{-5}{2}\right)x^{-7/2}$$

$$\frac{d^2y}{dx^2} = \frac{9}{4}x^{-5/2} - \frac{15}{2}x^{-7/2} \longrightarrow \frac{d^2y}{dx^2} = \frac{9}{4}(+2)^{-5/2} - \frac{15}{2}(2)^{-7/2}$$

$$d^2y/dx^2 = 2^{-7/2}\left(\frac{9}{4}(2)^1 - \frac{15}{2}\right) = -0.265$$

Since $d^2y/dx^2 < 0$ at $x = 2$, the stationary point in a maximum

(c) Find the area enclosed by the curve, the x-axis and the lines $x = k$ and $x = 4$.
Give your answer in the form $a + b\sqrt{c}$, where a, b and c are integers to be found. [4]



$A = \int_{2}^{4} \left(3x^{-1/2} - 2x^{-3/2}\right) dx$

$A = \left[6x^{1/2} + 4x^{-1/2} \right]_{2}^{4}$

$A = \left| 6(4)^{1/2} + 4(4)^{-1/2} \right| - \left| 6(2)^{1/2} + 4(2)^{-1/2} \right|$

$= 14 - 8\sqrt{2}$ where $a = 14$, $b = -8$, and $c = 2$

8 An arithmetic progression has first term 20 and common difference d. A geometric progression also has first term 20 and common ratio r, where $r > 0$.

The third term of the geometric progression is 5 more than the third term of the arithmetic progression. The fifth term of the geometric progression is 30 more than the fifth term of the arithmetic progression.

(a) Find the value of r and the value of d. [4]

$G_3 = A_3 + 5$ $\quad\quad$ $G_5 = A_5 + 30$

$20r^2 = (20 + 2d) + 5$ $\quad\quad$ $20r^4 = (20 + 4d) + 30$

$2d = 20r^2 - 25$ $\quad\quad$ $20r^4 = 50 + 4d$

$$20r^4 = 50 + \cancel{4}^{2}\left(\frac{20r^2 - 25}{\cancel{2}}\right)$$

$$20r^4 = 50 + 40r^2 - 50$$

$$20r^4 = 40r^2$$

$$r^2 = 2 \longrightarrow r = +\sqrt{2} \text{ only}$$

$$d = \frac{20(\sqrt{2})^2 - 25}{2} \longrightarrow d = 7.5$$

(b) Show that the ninth term of the geometric progression is 4 times the ninth term of the arithmetic progression. [2]

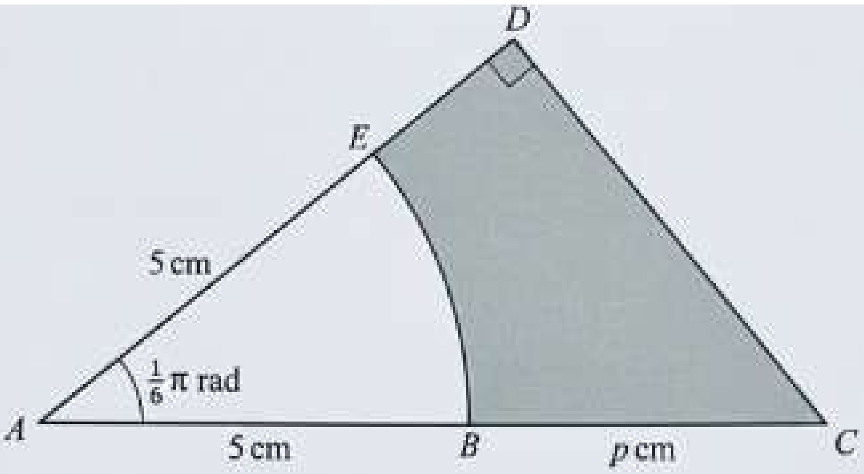
$G_n = ar^{n-1} \longrightarrow G_9 = 20(\sqrt{2})^{9-1} = 20 \times 16 = 320$

$A_9 = 20 + (9-1)(7.5) \longrightarrow \frac{G_9}{A_9} = \frac{320}{80} = 4$

$A_9 = 80$

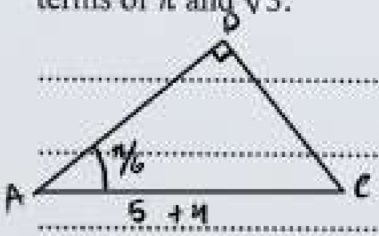
Therefore $G_9 = 4 \times A_9$

9



The diagram shows a triangle ACD in which AD is perpendicular to CD. The arc BE of a circle with centre A and radius 5 cm meets AC at B and AD at E. Angle BAE is $\frac{1}{6}\pi$ radians and the length $BC = p$ cm.

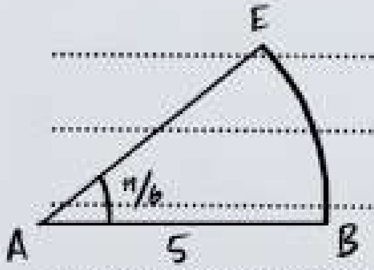
(a) Given that the value of p is 4, find the exact perimeter of the shaded region. Give your answer in terms of π and $\sqrt{3}$. [4]



$CD = AC\sin(\pi/6)$

$CD = 9\sin(\pi/6) = 4.5$ cm

$AD = 9\cos(\pi/6) = 9\sqrt{3}/2$ cm



$BE = 5 \times (\pi/6) = 5\pi/6$

Perimeter $= ED + CD + BC + BE$

$\left(\frac{9\sqrt{3}}{2} - 5\right) + 4.5 + 4 + \frac{5\pi}{6}$

Perimeter $= \left(\frac{7 + 9\sqrt{3}}{2} + \frac{5}{6}\pi\right)$ cm

(b) Given **instead** that the area of the shaded region is $8\sqrt{3} - \frac{25}{12}\pi \text{ cm}^2$, find the value of p.

$A_{\text{SHADED REGION}} = A_{\text{TRIANGLE}} - A_{\text{SECTOR}}$

$$8\sqrt{3} - \frac{25}{12}\pi = \left(\frac{1}{2} \times AD \times CD\right) - \left(\frac{1}{2} \times r^2 \times \theta\right)$$

$$8\sqrt{3} - \frac{25}{12}\pi = \frac{1}{2} \times \frac{AC\sqrt{3}}{2} \times \frac{AC}{2} - \frac{1}{2}(5)^2(\pi/6)$$

$$8\sqrt{3} - \cancel{\frac{25\pi}{12}} = \frac{\sqrt{3}\,AC^2}{8} - \cancel{\frac{25\pi}{12}}$$

$$\frac{\cancel{\sqrt{3}}}{8} AC^2 = 8\cancel{\sqrt{3}} \qquad AC^2 = 64 \longrightarrow AC = 8 \text{ only}$$

$$AC = 5 + p$$
$$8 = 5 + p \longrightarrow p = 3$$

10 Functions f and g are defined as follows:

$$f(x) = 3x - 6 \qquad \text{for } x > 0,$$

$$g(x) = \frac{4}{(ax-3)^2} \qquad \text{for } x > \frac{3}{a},$$

where a is a positive constant.

(a) State the range of f. [1]

$f(0) = 3(0) - 6 = -6$

Range: $f(x) > -6$

(b) Find $g^{-1}(x)$. [2]

$y = \frac{4}{(ax-3)^2}$ $\quad x > 3/a \rightarrow ax - 3 > 0$

$(ax-3)^2 = \frac{4}{y} \longrightarrow ax - 3 = \sqrt{\frac{4}{y}}$

$x = \left(\frac{2}{\sqrt{y}} + 3\right) \div a$

$g^{-1}(x) = \frac{1}{a}\left(\frac{2}{\sqrt{x}} + 3\right)$

(c) Given that $a = 2$, solve the equation $g^{-1}(x) = 4$. [2]

$4 = \frac{1}{2}\left(\frac{2}{\sqrt{x}} + 3\right)$

$(4 \times 2) - 3 = 2/\sqrt{x}$

$\sqrt{x} = 2/5$

$x = 4/25$

(d) Given instead that fg(8) = 6, find the value of a, justifying your answer. [4]

$g(8) = 4/(8a-3)^2$

$fg(8) = 3\left(\frac{4}{(8a-3)^2}\right) - 6 \longrightarrow 6 = \frac{12}{(8a-3)^2} - 6$

$\cancel{12} = \frac{\cancel{12}}{(8a-3)^2}$

$(8a-3)^2 = 1$

$8a-3 = \pm 1$

$8a-3 = 1$ or $8a-3 = -1$

$a = 1/2$ or $a = 1/4$

Domain if $a = 1/2$

$x > \frac{3}{1/2} \rightarrow x > 6$

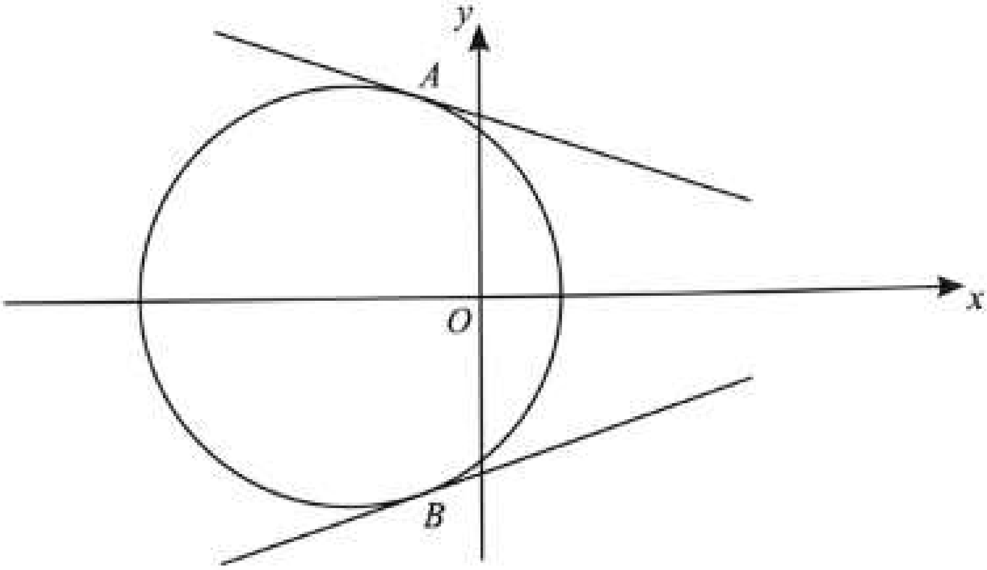
Domain if $a = 1/4$

$x > \frac{3}{1/4} \longrightarrow x > 12$

if $x > 12$, $g(8)$ is not valid.

$a = 1/2$ only

11



A tangent to a circle passes through the points (1, 5) and (4, 4) and meets the circle at the point A. Another tangent to the circle has equation $x - 3y = 16$ and meets the circle at the point B (see diagram).

(a) Find the coordinates of the point of intersection of the two tangents. [4]

Equation of tangent at A: $m = \frac{4-5}{4-1} = -\frac{1}{3}$

$y - 5 = -1/3\,(x - 1) \longrightarrow x + 3y = 16$

$x = 16 - 3y$

Sub in eq $x - 3y = 16$

$16 - 3y - 3y = 16 \rightarrow 3y = 0 \rightarrow y = 0$

$x = 16 - 3(0) \rightarrow 16$

Coordinates of the point of intersection is (16, 0)

It is given that the coordinates of the centre of the circle are $(a, 0)$, where $a < 0$.

(b) Find an equation of the normal to the circle which passes through A. Give your answer in terms of a. [2]

normal passes thru centre $(a, 0)$

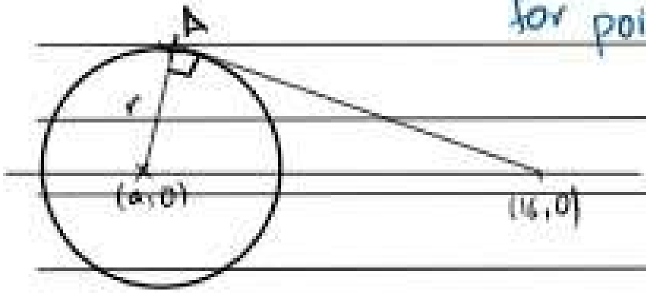
gradient of normal $= -1/(-1/3) = 3$

$y - 0 = 3(x - a)$

$y = 3x - 3a$

(c) It is further given that the radius of the circle is $\sqrt{40}$.

Find an equation of the circle. [4]



for point A: $x + 3(3x - 3a) = 16$

$x + 9x - 9a = 16$

$x = (16 + 9a)/10$

$y = \frac{3(16 + 9a)}{10} - 3a$

$y = \frac{48 - 3a}{10}$

radius $= \sqrt{40} = \sqrt{\left(\frac{16 + 9a}{10} - a\right)^2 + \left(\frac{48 - 3a}{10} - 0\right)^2}$

$40 = \frac{(16 - a)^2 + (3(16 - a))^2}{100}$

$40 = \frac{\cancel{10}(16 - a)^2}{10\cancel{0}}$

$400 = (16 - a)^2 \longrightarrow 16 - a = \pm 20$

$a = -4$ (accept)

$(x - (-4))^2 + (y - 0)^2 = (\sqrt{40})^2$

Eq of circle: $(x + 4)^2 + y^2 = 40$