1 Solve the simultaneous equations

$$4x^2 + 2y^2 - 3y = 21 \qquad \text{and} \qquad 2x - y - 5 = 0.$$

[5]

$2x = y + 5$

$(y+5)^2 + 2y^2 - 3y = 21$

$y^2 + 10y + 25 + 2y^2 - 3y - 21 = 0$

$3y^2 + 7y + 4 = 0$

$(3y+4)(y+1) = 0 \longrightarrow y = -1$ or $y = -4/3$

- When $y = -1$

$2x = -1 + 5 \longrightarrow x = 2$

- When $y = -4/3$

$2x = -4/3 + 5 \longrightarrow x = 11/6$

$(2, -1)$ and $(11/6, -4/3)$

2 A tank is being filled with water at a rate of $200\,\text{cm}^3\,\text{s}^{-1}$. The volume of water in the tank, $V\,\text{cm}^3$, is given by

$$V = 100\pi h + \frac{1}{3}\pi h^3,$$

where h is the height of water in the tank in centimetres.

Find the exact rate at which h is increasing when $h = 30$. [3]

$\frac{dV}{dh} = \frac{d}{dh}\left(100\pi h + \frac{1}{3}\pi h^3\right) \longrightarrow 100\pi + \pi h^2$

dV/dh at $h = 30 \longrightarrow 100\pi + \pi(30)^2 = 1000\pi$

$\frac{dh}{dt} = \frac{dV/dt}{dV/dh} = \frac{200}{1000\pi} = \frac{1}{5\pi}\ \text{cm s}^{-1}$

3 The coefficient of x^4 in the expansion of $(p+2x)^5$ is equal to the coefficient of x^2 in the expansion of $(1-2qx)^4$, where p and q are constants.

(a) Find an expression for p in terms of q. [3]

$^5C_4(p)^{5-4}(2)^4 = {}^4C_2(1)^{4-2}(-2q)^2$

$p = \dfrac{^4C_2 \times 4q^2}{^5C_4 \times 16}$

$p = \dfrac{\cancel{6}^{3} \times 4q^2}{5 \times \cancel{16}^{8 \times 2}}$

$p = \dfrac{3}{10}q^2$

(b) Given that $p = \frac{6}{5}$, find the two possible values of the coefficient of x^3 in the expansion of $(3+qx)^6$. [3]

$6/5 = (3/10)\,q^2$

$q^2 = \dfrac{{}^2\cancel{6}/\cancel{5}^1}{{}^1\cancel{3}/\cancel{10}^2} = 4 \longrightarrow q = \pm 2$

Coefficient of x^3 in $(3+qx)^6$

$^6C_3(3)^3(q)^3$

$20 \times 27 \times (\pm 2)^3 = \pm 4320$

two possible values of the coefficient of x^3 are $+4320$ and -4320

4 The equation of a curve is $y = -4x^2 + 2x + 3$.

(a) Use differentiation to find the x-coordinate of the stationary point of the curve. [2]

$\frac{dy}{dx} = -8x + 2$

$dy/dx = 0$ at SP, $-8x + 2 = 0$

$x = 1/4$

The curve is stretched by factor 3 in the y-direction and then by factor $\frac{1}{2}$ in the x-direction.

(b) Find the x-coordinate of the stationary point of the transformed curve. [1]

$x_{new} = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

(c) Find the equation of the transformed curve. Give your answer in the form $y = ax^2 + bx + c$, where a, b and c are integers to be found. [3]

$y = 3[-4(2x)^2 + 2(2x) + 3]$

$y = 3[-16x^2 + 4x + 3]$

$y = -48x^2 + 12x + 9$ where $a = -48$, $b = 12$ and $c = 9$

5 (a) Express $2x^2 - 16x - 18$ in the form $2(x+a)^2 + b$, where a and b are constants to be found. [2]

T.P. $\left(-b/2a, f(-b/2a)\right)$ $\quad -b/2a = -(-16)/2(2) = 4$

$f(4) = 2(4)^2 - 16(4) - 18 = -50$

$y = a(x-h)^2 + k$

$y = 2(x-4)^2 - 50$

(b) Hence, solve the equation $2x^8 - 16x^4 - 18 = 0$. [4]

$2(x^4 - 4)^2 - 50 = 0$

$(x^4 - 4)^2 = 25$

$x^4 - 4 = +5$ or $x^4 - 4 = -5$

$x^4 = 9$ $\quad x^4 = -1$

$x = \pm\sqrt{\sqrt{9}}$ $\quad$ no real solutions

$x = \pm\sqrt{3}$

6 Find the set of values of k for which the line $y + 4x = k$ and the curve with equation $y = 4 - 7x - 2x^2$ have at least one point of intersection. [4]

$k - 4x = 4 - 7x - 2x^2$

$2x^2 + 3x + (k - 4) = 0$

For at least one point of intersection, $\Delta \geq 0$

$(3)^2 - 4(2)(k-4) \geq 0$

$9 - 8k + 32 \geq 0$

$8k \leq 41$

$k \leq \frac{41}{8}$

7 (a) Prove the identity

$$\frac{\cos\theta}{1-\sin\theta} - \frac{1}{\cos\theta} = \tan\theta. \quad [4]$$

$$\frac{\cos\theta(\cos\theta) - 1(1-\sin\theta)}{(1-\sin\theta)(\cos\theta)}$$

$$\frac{\cos^2\theta - 1 + \sin\theta}{(1-\sin\theta)(\cos\theta)} \quad ; \cos^2\theta + \sin^2\theta = 1$$

$$\frac{1-\sin^2\theta - 1 + \sin\theta}{(1-\sin\theta)(\cos\theta)} \longrightarrow \frac{\sin\theta\cancel{(1-\sin\theta)}}{\cancel{(1-\sin\theta)}(\cos\theta)}$$

$$= \frac{\sin\theta}{\cos\theta} = \boxed{\tan\theta}$$

(b) Hence, solve the equation

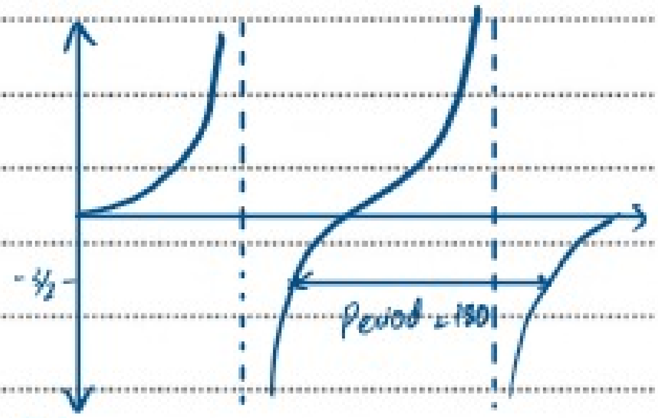
$$\frac{2\cos\theta}{1-\sin\theta} - \frac{2}{\cos\theta} + 3 = 0$$

for $0° \leqslant \theta \leqslant 360°$. [3]

$$2\left(\frac{\cos\theta}{1-\sin\theta} - \frac{1}{\cos\theta}\right) + 3 = 0 \longrightarrow 2\tan\theta + 3 = 0$$

$\tan\theta = -3/2$

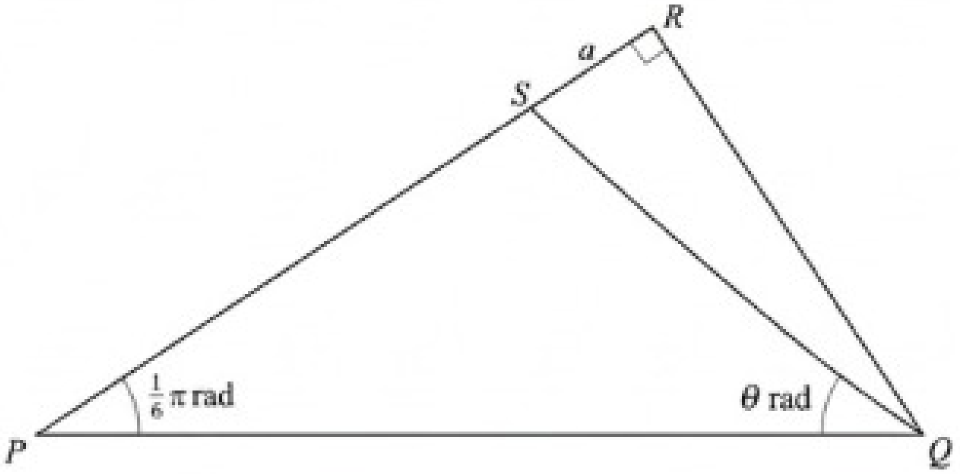
$\alpha = \tan^{-1}(3/2) \simeq 56.31°$



$\theta = 180 - 56.31,\ 360 - 56.31$

$= 123.7°,\ 303.7°$

8

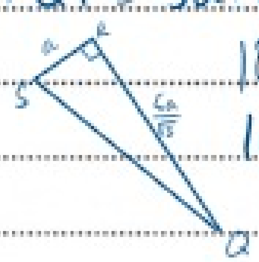


The diagram shows a right-angled triangle PQR. The point S lies on PR and is such that $PS = 4RS$. The distance $RS = a$. The angle $QPR = \frac{1}{6}\pi$ radians, and the angle $PQS = \theta$ radians.

(a) Find the exact distance QS in terms of a. Give your answer in the form $\sqrt{\frac{c}{d}}a$, where c and d are integers to be found. [2]

$|PS| = 4a$

$|RQ| = 5a\tan(\pi/6) = 5a/\sqrt{3}$



$|QS| = \sqrt{(a)^2 + (5a/\sqrt{3})^2}$

$|QS| = \sqrt{a^2 + 25a^2/3}$

$|QS| = \sqrt{\frac{28}{3}} \cdot a$

(b) (i) Show that the angle $RQS = \sin^{-1}\left(\frac{\sqrt{21}}{14}\right)$. [2]

$\sin(\angle RQS) = \frac{a}{a\sqrt{28/3}} = \frac{\sqrt{3}}{2\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{21}}{14}$

angle $RQS = \sin^{-1}\left(\frac{\sqrt{21}}{14}\right)$

(ii) Hence, find the exact value of θ. [1]

$\angle RQP = \pi/2 - \pi/6 = \pi/3$

$\theta = \pi/3 - \sin^{-1}\left(\frac{\sqrt{21}}{14}\right)$

9 A function f is defined by $f(x) = 3 - \dfrac{4}{2x+3}$ for $x > -\frac{3}{2}$.

(a) (i) Find $f'(x)$. [2]

$f(x) = 3 - 4(2x+3)^{-1}$

$f'(x) = 0 - 4(-1)(2x+3)^{-2} \cdot \dfrac{d}{dx}(2x+3)$

$f'(x) = \dfrac{8}{(2x+3)^2}$

(ii) Hence, show that f is an increasing function. [1]

The denominator $(2x+3)^2$ is positive for $x > -3/2$

$f'(x) > 0$ for the given domain

Therefore, f is an increasing function

(b) (i) Find an expression for $f^{-1}(x)$. [3]

$y = 3 - \dfrac{4}{2x+3} \longrightarrow \dfrac{4}{2x+3} = 3 - y$

$\dfrac{4}{3-y} = 2x+3 \longrightarrow x = \dfrac{\frac{4}{3-y} - 3}{2}$

$x = \dfrac{2}{3-y} - \dfrac{3}{2}$

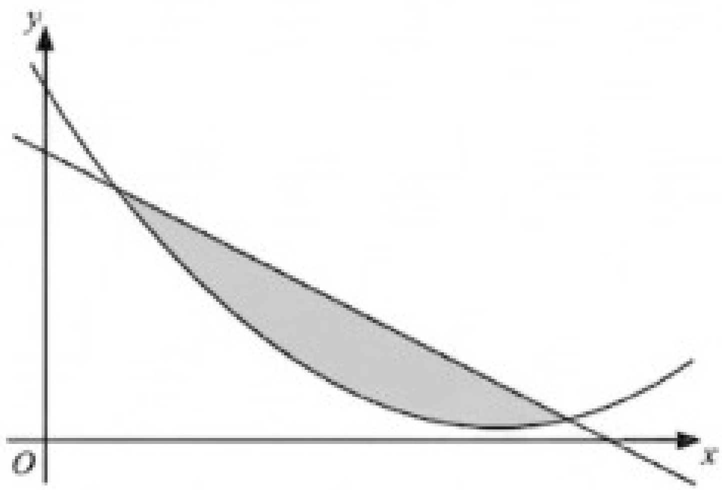
$f^{-1}(x) = \dfrac{2}{3-x} - \dfrac{3}{2}$

(ii) State the domain and range of f^{-1}. [2]

Range: $f^{-1}(x) > -3/2$

Domain: $x < 3$

10



The diagram shows the curve with equation $y = x^2 - 12x + 37$ and the line $y = -4x + 30$.

Find the area of the shaded region. [7]

$y_{new} = y_{top} - y_{bottom}$

$y_{new} = -4x + 30 - (x^2 - 12x + 37) = -x^2 + 8x - 7$

$-x^2 + 8x - 7 = 0$

$-(x-1)(x-7) = 0$

$x = 1, \quad x = 7$

$\text{Area} = \int_1^7 (-x^2 + 8x - 7)\,dx$

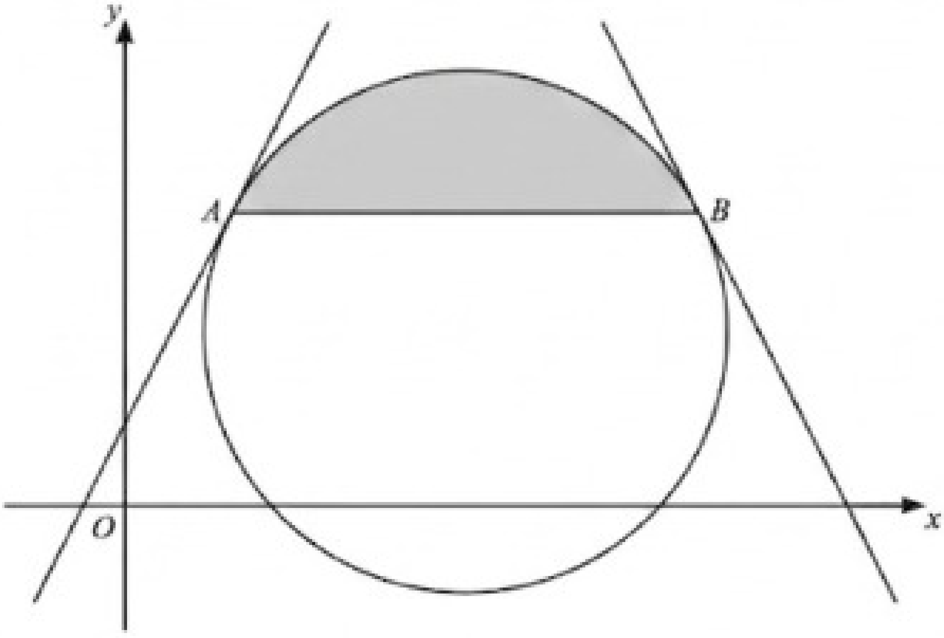
$= \left[\frac{-x^3}{3} + 4x^2 - 7x \right]_1^7$

$= \left(-7^3/3 + 4(7)^2 - 7(7)\right) - \left(-1^3/3 + 4(1)^2 - 7(1)\right)$

$= (98/3) - (-10/3)$

Area $= 36$

11



In the diagram, AB is a chord of the circle with equation $(x-6)^2+(y-3)^2=20$. The two tangents to the circle with equations $y=2x+1$ and $y=-2x+25$ meet the circle at the points A and B respectively.

(a) Find the coordinates of the points A and B. [4]

$(x-6)^2+(2x+1-3)^2=20$

$x^2-12x+36+4x^2-8x+4=20$

$5x^2-20x+20=0$

$x^2-4x+4=0 \longrightarrow (x-2)^2=0$

$x=+2$

$y=2(+2)+1=+5$

$(x-6)^2+(-2x+25-3)^2=20$

$x^2-12x+36+4x^2-88x+484=20$

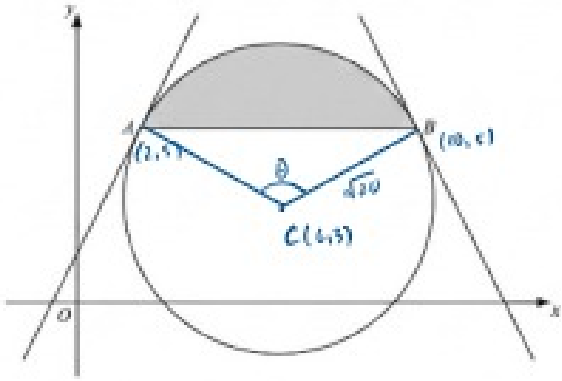
$5x^2-100x+500=0 \longrightarrow (x-10)^2=0$

$x=+10$

$y=-2(+10)+25=+5$

$A(2,5)$ and $B(10,5)$

(b) Hence, or otherwise, find the area of the shaded region. Give your answer to 3 significant figures. [5]



Centre is $(6,3)$

radius is $\sqrt{20}$

$|AB|$ is 10

$\cos\theta = \dfrac{8^2 - [(\sqrt{20})^2 + (\sqrt{20})^2]}{-2(\sqrt{20})(\sqrt{20})}$

$\cos\theta = -3/5$

$\theta = 2.214$ rad

$A_{SHADED} = A_{SECTOR} - A_{TRIANGLE}$

$= \dfrac{1}{2}(\sqrt{20})^2(2.214) - \dfrac{1}{2}(\sqrt{20})^2 \overbrace{(4/5)}^{\sin\theta}$

$= 22.143 - 8$

$A_{SHADED} = 14.1$ to 3sf

12 (a) The first term of an arithmetic progression is 5730 and the common difference is –23.

Find the value of the first negative term. [3]

$5730 + (n-1)(-23) < 0$

$5730 - 23n + 23 < 0$

$n > \frac{5753}{23} \approx 250.13$

$u_{251} = 5730 + (251-1)(-23)$

$= -20$

first negative term is -20

(b) The first term of a geometric progression is $\sin\theta$ and the second term is $\frac{1}{2}\tan\theta$, where $0 < \theta < \frac{1}{2}\pi$.

(i) Find the set of values of θ for which the progression is convergent. [3]

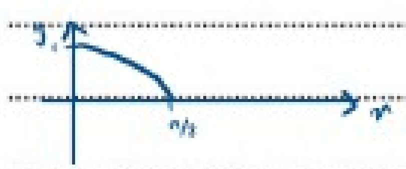
$r = \frac{(1/2)\tan\theta}{\sin\theta} = \frac{(1/2)(\sin\theta/\cos\theta)}{\sin\theta} = \frac{1}{2\cos\theta}$

$\left|\frac{1}{2\cos\theta}\right| < 1 \longrightarrow |\cos\theta| > 1/2$

$\cos\theta > 1/2$ or $\cos\theta < -1/2$

$\cos\theta$ is +ve in the given domain so no solutions

$0 < \theta < \pi/3$



(ii) Given that $\theta = \frac{1}{6}\pi$, find the sum to infinity of the progression. Give your answer in the form $\frac{p+\sqrt{q}}{r}$, where p, q and r are integers to be found. [3]

$r = \dfrac{1}{2\cos(\pi/6)} = \dfrac{1}{\sqrt{3}}$

$a = \sin(\pi/6) = 1/2$

$S_\infty = \dfrac{\frac{1}{2}}{1 - \frac{1}{\sqrt{3}}} = \dfrac{1}{2} \times \dfrac{\sqrt{3}}{\sqrt{3}-1}$

$= \dfrac{\sqrt{3}(\sqrt{3}+1)}{2(\sqrt{3}-1)(\sqrt{3}+1)}$

$S_\infty = \dfrac{3+\sqrt{3}}{4}$