

A-Level Mathematics Paper Solutions

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1. Find the quotient and remainder when $4x^3 - 4x^2 + 7x + 2$ is divided by $2x - 1$. [3]

Solution

We can use algebraic long division to divide the polynomial $4x^3 - 4x^2 + 7x + 2$ by $2x - 1$:

$$(4x^3 - 4x^2 + 7x + 2) \div (2x - 1)$$

1. Divide the leading term: $4x^3/2x = 2x^2$.
 2. Multiply and subtract: $(4x^3 - 4x^2) - (4x^3 - 2x^2) = -2x^2$.
 3. Bring down the next term to get $-2x^2 + 7x$.
 4. Divide again: $-2x^2/2x = -x$.
 5. Multiply and subtract: $(-2x^2 + 7x) - (-2x^2 + x) = 6x$.
 6. Bring down the final term to get $6x + 2$.
 7. Divide: $6x/2x = 3$.
 8. Multiply and subtract: $(6x + 2) - (6x - 3) = 5$.
- This leaves us with a quotient of $2x^2 - x + 3$ and a remainder of 5.
- Quotient: $2x^2 - x + 3$, Remainder: 5**

2. Solve the equation $2^{5-2x} = \frac{4^{x+1}}{3}$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers. [4]

Solution

First, express all exponential terms with the base 2 and rearrange the equation to isolate the constant 3:

$$\begin{aligned} 3 \cdot 2^{5-2x} &= (2^2)^{x+1} \\ 3 \cdot 2^{5-2x} &= 2^{2x+2} \\ 3 &= \frac{2^{2x+2}}{2^{5-2x}} \\ 3 &= 2^{(2x+2)-(5-2x)} \\ 3 &= 2^{4x-3} \end{aligned}$$

Now, take the natural logarithm of both sides:

$$\ln 3 = \ln(2^{4x-3})$$

$$\ln 3 = (4x - 3) \ln 2$$

$$\ln 3 = 4x \ln 2 - 3 \ln 2$$

$$4x \ln 2 = \ln 3 + 3 \ln 2$$

$$x = \frac{\ln 3 + \ln(2^3)}{4 \ln 2}$$

$$x = \frac{\ln 3 + \ln 8}{\ln(2^4)}$$

$$x = \frac{\ln 24}{\ln 16}$$

$$\frac{\ln 24}{\ln 16}$$

3. The complex numbers s and t are given by $s = 3(\cos 2 + i \sin 2)$ and $t = 2e^{-i}$.
- (a) Express st in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$ and $r > 0$. [2]
- (b) In an Argand diagram with origin O, the points A and B represent the complex numbers s and st respectively. By considering the line segments OA and OB, or otherwise, state the two geometric effects on OA of multiplying s by t . [2]

Solution

(a) First, write s in exponential form: $s = 3e^{2i}$.

Now, calculate the product st :

$$st = (3e^{2i})(2e^{-i})$$

$$st = (3 \times 2)e^{i(2-1)}$$

$$st = 6e^i$$

$$6e^i$$

(b) The complex number $t = 2e^{-i}$ has a modulus of 2 and an argument of -1 radian. Multiplying by t applies these properties geometrically to the line segment OA:

- The length of OA is multiplied by the modulus of t , meaning an enlargement.
- The direction of OA is rotated by the argument of t .

An enlargement by a scale factor of 2 centered at the origin O, and a clockwise rotation of 1 radian about the origin O.

4. The square roots of $3 - (6\sqrt{2})i$ can be expressed in the Cartesian form $x + iy$, where x and y are real and exact. By first forming a quartic equation, find the square roots of $3 - (6\sqrt{2})i$. Give your answer in exact Cartesian form. [5]

Solution

Let $(x + iy)^2 = 3 - 6\sqrt{2}i$. Expanding the left-hand side yields:

$$x^2 - y^2 + 2xyi = 3 - 6\sqrt{2}i$$

Equating the real and imaginary parts:

$$x^2 - y^2 = 3 \tag{1}$$

$$2xy = -6\sqrt{2} \implies x = -\frac{3\sqrt{2}}{y} \tag{2}$$

Substitute equation (2) into equation (1) to form a quartic in terms of y :

$$\begin{aligned} \left(-\frac{3\sqrt{2}}{y}\right)^2 - y^2 &= 3 \\ \frac{18}{y^2} - y^2 &= 3 \\ 18 - y^4 &= 3y^2 \\ y^4 + 3y^2 - 18 &= 0 \end{aligned}$$

Factoring the quartic equation (treating it as a quadratic in y^2):

$$(y^2 + 6)(y^2 - 3) = 0$$

Since y must be a real number, $y^2 \geq 0$. We reject $y^2 = -6$, which leaves:

$$y^2 = 3 \implies y = \pm\sqrt{3}$$

Now, substitute y back into $x = -\frac{3\sqrt{2}}{y}$ to find the corresponding x values:

- If $y = \sqrt{3}$, then $x = -\frac{3\sqrt{2}}{\sqrt{3}} = -\sqrt{6}$.
- If $y = -\sqrt{3}$, then $x = -\frac{3\sqrt{2}}{-\sqrt{3}} = \sqrt{6}$.

Therefore, the square roots are $-\sqrt{6} + i\sqrt{3}$ and $\sqrt{6} - i\sqrt{3}$.

$$\pm(\sqrt{6} - i\sqrt{3})$$

5. The parametric equations of a curve are $x = 2\theta - \tan 2\theta$, $y = 6\theta + 2 \ln(\cos 2\theta)$, for $0 \leq \theta \leq \frac{1}{4}\pi$.

(a) Find an expression for $\frac{dy}{dx}$ in terms of $\tan 2\theta$. [4]

(b) Find the exact coordinates of the point on the curve at which the gradient of the tangent is equal to -1. [6]

Solution

(a) Differentiate x and y with respect to θ . Using the identity $\sec^2 A = 1 + \tan^2 A$:

$$\begin{aligned}\frac{dx}{d\theta} &= 2 - 2\sec^2(2\theta) = 2(1 - \sec^2(2\theta)) = -2\tan^2(2\theta) \\ \frac{dy}{d\theta} &= 6 + 2 \cdot \frac{1}{\cos 2\theta} \cdot (-2 \sin 2\theta) = 6 - 4\tan(2\theta)\end{aligned}$$

Apply the chain rule to find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{6 - 4\tan(2\theta)}{-2\tan^2(2\theta)} = \frac{2\tan(2\theta) - 3}{\tan^2(2\theta)}$$

$$\frac{2\tan 2\theta - 3}{\tan^2 2\theta}$$

(b) Set the gradient to -1 and solve for θ :

$$\begin{aligned}\frac{2\tan(2\theta) - 3}{\tan^2(2\theta)} &= -1 \\ 2\tan(2\theta) - 3 &= -\tan^2(2\theta) \\ \tan^2(2\theta) + 2\tan(2\theta) - 3 &= 0 \\ (\tan(2\theta) - 1)(\tan(2\theta) + 3) &= 0\end{aligned}$$

Since $0 \leq \theta \leq \frac{\pi}{4}$, the range for 2θ is $[0, \frac{\pi}{2}]$, so $\tan(2\theta)$ must be positive. Thus, $\tan(2\theta) = 1 \implies 2\theta = \frac{\pi}{4} \implies \theta = \frac{\pi}{8}$.
Substitute $\theta = \frac{\pi}{8}$ into the parametric equations:

$$\begin{aligned}x &= 2\left(\frac{\pi}{8}\right) - \tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - 1 \\ y &= 6\left(\frac{\pi}{8}\right) + 2\ln\left(\cos\frac{\pi}{4}\right) = \frac{3\pi}{4} + 2\ln(2^{-1/2}) = \frac{3\pi}{4} - \ln 2\end{aligned}$$

$$\left(\frac{\pi}{4} - 1, \frac{3\pi}{4} - \ln 2\right)$$

6. The equation of a curve is $y = x \tan^{-1} 3x$ for $-\pi \leq x \leq \pi$.

(a) Find the exact value of the gradient of the curve when $x = \frac{1}{3}$. [4]

(b) Find the exact value of $\int_0^{\frac{1}{3}} y dx$. [6]

Solution

(a) Use the product rule to differentiate $y = x \tan^{-1}(3x)$:

$$\begin{aligned}\frac{dy}{dx} &= 1 \cdot \tan^{-1}(3x) + x \cdot \frac{d}{dx}(\tan^{-1}(3x)) \\ \frac{dy}{dx} &= \tan^{-1}(3x) + x \cdot \frac{3}{1 + (3x)^2} \\ \frac{dy}{dx} &= \tan^{-1}(3x) + \frac{3x}{1 + 9x^2}\end{aligned}$$

Substitute $x = \frac{1}{3}$:

$$\left. \frac{dy}{dx} \right|_{x=1/3} = \tan^{-1} \left(3 \left(\frac{1}{3} \right) \right) + \frac{3(1/3)}{1+9(1/3)^2} = \tan^{-1}(1) + \frac{1}{1+1} = \frac{\pi}{4} + \frac{1}{2}$$

$$\frac{\pi}{4} + \frac{1}{2}$$

(b) Evaluate the integral $I = \int_0^{1/3} x \tan^{-1}(3x) dx$ using integration by parts.
Let $u = \tan^{-1}(3x)$, then $du = \frac{3}{1+9x^2} dx$.

Let $dv = x dx$, then $v = \frac{x^2}{2}$.

$$I = \left[\frac{x^2}{2} \tan^{-1}(3x) \right]_0^{1/3} - \frac{3}{2} \int_0^{1/3} \frac{x^2}{1+9x^2} dx$$

To solve the remaining integral, rewrite the numerator to separate the rational function:

$$\begin{aligned} \frac{3}{2} \int_0^{1/3} \frac{x^2}{1+9x^2} dx &= \frac{3}{2} \cdot \frac{1}{9} \int_0^{1/3} \frac{9x^2 + 1 - 1}{1+9x^2} dx \\ &= \frac{1}{6} \int_0^{1/3} \left(1 - \frac{1}{1+9x^2} \right) dx \\ &= \frac{1}{6} \left[x - \frac{1}{3} \tan^{-1}(3x) \right]_0^{1/3} \end{aligned}$$

Now compile all the terms evaluated at the bounds:

$$\begin{aligned} I &= \left(\frac{1}{18} \tan^{-1}(1) - 0 \right) - \frac{1}{6} \left[\left(\frac{1}{3} - \frac{1}{3} \tan^{-1}(1) \right) - 0 \right] \\ I &= \frac{1}{18} \left(\frac{\pi}{4} \right) - \frac{1}{18} + \frac{1}{18} \left(\frac{\pi}{4} \right) \\ I &= \frac{2\pi}{72} - \frac{1}{18} = \frac{\pi}{36} - \frac{2}{36} = \frac{\pi-2}{36} \end{aligned}$$

$$\frac{\pi-2}{36}$$

7. The equation of a curve is $y = \frac{3+\sin 3x}{2-\cos 3x}$ for $0 \leq x \leq \pi$. The gradient of the curve is equal to $-\frac{3}{2}$ when $x = a$.

(a) Show that a satisfies the equation $\sin^2 3a + 6 \sin 3a - 3 = 0$. [4]

(b) Hence find the values of a for $0 \leq a \leq \pi$. [3]

Solution

(a) Let $u = 3x$ to simplify the differentiation. $y = \frac{3+\sin u}{2-\cos u}$.

Using the quotient rule:

$$\frac{dy}{du} = \frac{(\cos u)(2 - \cos u) - (3 + \sin u)(\sin u)}{(2 - \cos u)^2}$$

$$\frac{dy}{du} = \frac{2 \cos u - \cos^2 u - 3 \sin u - \sin^2 u}{(2 - \cos u)^2} = \frac{2 \cos u - 3 \sin u - 1}{(2 - \cos u)^2}$$

Reapplying the chain rule for x : $\frac{dy}{dx} = 3 \cdot \frac{dy}{du}$.

At $x = a$, the gradient is $-3/2$:

$$3 \left(\frac{2 \cos 3a - 3 \sin 3a - 1}{(2 - \cos 3a)^2} \right) = -\frac{3}{2}$$

$$\frac{2 \cos 3a - 3 \sin 3a - 1}{(2 - \cos 3a)^2} = -\frac{1}{2}$$

$$-4 \cos 3a + 6 \sin 3a + 2 = (2 - \cos 3a)^2$$

$$-4 \cos 3a + 6 \sin 3a + 2 = 4 - 4 \cos 3a + \cos^2 3a$$

$$6 \sin 3a - 2 = \cos^2 3a$$

Using the identity $\cos^2 3a = 1 - \sin^2 3a$:

$$6 \sin 3a - 2 = 1 - \sin^2 3a$$

$$\sin^2 3a + 6 \sin 3a - 3 = 0$$

(Shown)

(b) Treat the equation as a quadratic in $X = \sin 3a$: $X^2 + 6X - 3 = 0$.

$$X = \frac{-6 \pm \sqrt{36 - 4(1)(-3)}}{2} = \frac{-6 \pm \sqrt{48}}{2} = -3 \pm 2\sqrt{3}$$

Since $-1 \leq \sin 3a \leq 1$, we reject $-3 - 2\sqrt{3}$. Thus, $\sin 3a = -3 + 2\sqrt{3} \approx 0.4641$.

The interval for $3a$ is $0 \leq 3a \leq 3\pi$.

Principal value: $3a_1 = \sin^{-1}(-3 + 2\sqrt{3}) \approx 0.4826$.

Other values in the domain $[0, 3\pi]$:

$$3a_2 = \pi - 0.4826 \approx 2.6590$$

$$3a_3 = 2\pi + 0.4826 \approx 6.7658$$

$$3a_4 = 3\pi - 0.4826 \approx 8.9422$$

Dividing by 3 gives the values of a :

$a = 0.161, 0.886, 2.26, 2.98$ (to 3 s.f.)

8.

(a) By sketching a suitable pair of graphs, show that the equation $\csc \frac{1}{2}x = \frac{1}{2}(x + 3)$ has exactly one root in the interval $0 \leq x \leq \pi$. [2]

(b) Show by calculation that this root lies between 1 and 1.2. [2]

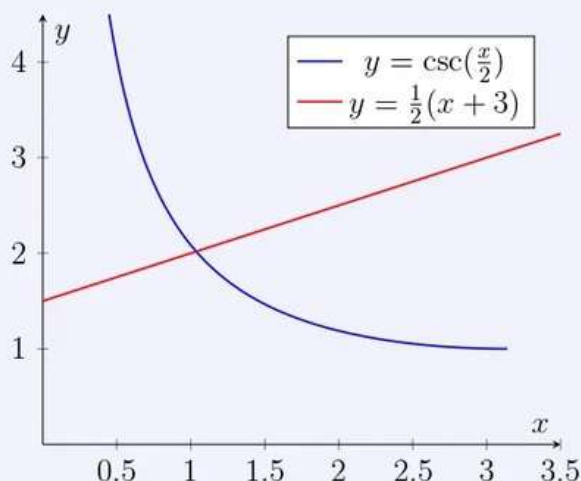
(c) Show that, if a sequence of real values given by the iterative formula $x_{n+1} = 2 \sin^{-1} \left(\frac{2}{x_n + 3} \right)$

converges, then it converges to the root of the equation in 8(a). [2]

(d) Use the iterative formula in 8(c) to calculate this root correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

Solution

(a) We sketch $y = \csc(\frac{x}{2})$ and the straight line $y = \frac{1}{2}(x + 3)$ for $x \in (0, \pi]$.



The two lines intersect exactly once in the interval, indicating one root.

(Shown via sketch)

(b) Let $g(x) = \csc(\frac{x}{2}) - \frac{1}{2}x - 1.5$. Evaluate at the boundaries:

$$g(1) = \csc(0.5) - 0.5(1) - 1.5 \approx 2.0858 - 2 = 0.0858 > 0$$

$$g(1.2) = \csc(0.6) - 0.5(1.2) - 1.5 \approx 1.7710 - 2.1 = -0.3290 < 0$$

Because $g(1)$ and $g(1.2)$ have opposite signs and the function is continuous, there is a root in $[1, 1.2]$.

Continuous function with $g(1) > 0$ and $g(1.2) < 0$; the sign change indicates a root lies between 1 and 1.2.

(c) Let the limit of the sequence be x .

$$x = 2 \sin^{-1} \left(\frac{2}{x+3} \right)$$

$$\frac{x}{2} = \sin^{-1} \left(\frac{2}{x+3} \right)$$

$$\sin \left(\frac{x}{2} \right) = \frac{2}{x+3}$$

$$\frac{1}{\sin(\frac{x}{2})} = \frac{x+3}{2}$$

$$\csc \left(\frac{x}{2} \right) = \frac{1}{2}(x+3)$$

This maps perfectly onto the original equation.

(Shown)

(d) Utilizing $x_{n+1} = 2 \sin^{-1} \left(\frac{2}{x_n+3} \right)$ with $x_0 = 1.1$:

$$x_1 = 1.01915$$

$$x_2 = 1.04170$$

$$x_3 = 1.03530$$

$$x_4 = 1.03711$$

$$x_5 = 1.03660$$

$$x_6 = 1.03674$$

$$x_7 = 1.03670$$

$$x_8 = 1.03671$$

The values stabilize. Correct to 3 decimal places, the root is 1.037.

1.037

9. The points A and B have position vectors $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$ respectively. The line m passes through the points A and B.

(a) Find a vector equation for m . [2]

The line l has equation $r = \begin{pmatrix} 6 \\ -1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$.

(b) Determine whether the lines l and m are parallel, intersect or are skew. [6]

(c) The point P lies on l and is such that angle $PAB = 60^\circ$. Find the position vector of P. [4]

Solution

(a) The direction vector of m is $\vec{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 4-3 \\ 3-4 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Using point A for the position vector:

$$\vec{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(b) The direction vectors $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ are not scalar multiples, so the lines are **not parallel**.

To test for intersection, equate the components:

$$3 + \lambda = 6 + \mu \quad (3)$$

$$4 - \lambda = -1 - \mu \quad (4)$$

$$1 = -3 - 2\mu \implies 2\mu = -4 \implies \mu = -2 \quad (5)$$

Substitute $\mu = -2$ into equation (1):

$$3 + \lambda = 6 - 2 = 4 \implies \lambda = 1$$

Now, check these values in equation (2):

$$\text{LHS} = 4 - 1 = 3 \quad \text{and} \quad \text{RHS} = -1 - (-2) = 1$$

Since $3 \neq 1$, there is a contradiction. The lines do not intersect.

Skew

(c) Since P lies on line l , $\vec{OP} = \begin{pmatrix} 6 + \mu \\ -1 - \mu \\ -3 - 2\mu \end{pmatrix}$. The vector $\vec{AP} = \vec{OP} - \vec{OA} = \begin{pmatrix} 3 + \mu \\ -5 - \mu \\ -4 - 2\mu \end{pmatrix}$.

We use the dot product identity: $\vec{AP} \cdot \vec{AB} = |\vec{AP}| |\vec{AB}| \cos(60^\circ)$

$$\vec{AP} \cdot \vec{AB} = (3 + \mu)(1) + (-5 - \mu)(-1) + (-4 - 2\mu)(0) = 8 + 2\mu$$

$$|\vec{AB}| = \sqrt{1^2 + (-1)^2 + 0} = \sqrt{2}$$

$$|\vec{AP}|^2 = (3 + \mu)^2 + (-5 - \mu)^2 + (-4 - 2\mu)^2 = 6\mu^2 + 32\mu + 50$$

Substitute into the dot product formula:

$$8 + 2\mu = \sqrt{6\mu^2 + 32\mu + 50} \cdot \sqrt{2} \cdot \frac{1}{2}$$

$$8 + 2\mu = \sqrt{3\mu^2 + 16\mu + 25}$$

Square both sides:

$$64 + 32\mu + 4\mu^2 = 3\mu^2 + 16\mu + 25$$

$$\mu^2 + 16\mu + 39 = 0$$

$$(\mu + 3)(\mu + 13) = 0 \implies \mu = -3 \text{ or } \mu = -13$$

Check for the acute angle 60° (dot product $8 + 2\mu$ must be positive):

- If $\mu = -13$, $8 + 2(-13) = -18 < 0$ (Reject).
- If $\mu = -3$, $8 + 2(-3) = 2 > 0$ (Accept).

Substitute $\mu = -3$ back into $\vec{OP}$:

$$\vec{OP} = \begin{pmatrix} 6 - 3 \\ -1 - (-3) \\ -3 - 2(-3) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$

10.

(a) Express $\frac{x+3}{(x-1)(2x-6)}$ in partial fractions. [4]

(b) The variables x and y satisfy the differential equation $(x-1)(2x-6)\frac{dy}{dx} = \frac{x+3}{\ln(2y)}$. It is given that $y = 1$ when $x = 4$. Solve the differential equation to obtain an equation in x and y . [7]

Solution

(a) Write the expression as:

$$\frac{x+3}{(x-1)(2x-6)} = \frac{A}{x-1} + \frac{B}{2x-6}$$

Multiply by the denominator:

$$x+3 = A(2x-6) + B(x-1)$$

Substitute smart values for x to find A and B :

- Let $x = 1 \implies 1 + 3 = A(2 - 6) \implies 4 = -4A \implies A = -1$.
- Let $x = 3 \implies 3 + 3 = B(3 - 1) \implies 6 = 2B \implies B = 3$.

$$\frac{3}{2x-6} - \frac{1}{x-1}$$

(b) Separate the variables in the differential equation:

$$\begin{aligned} \ln(2y) dy &= \frac{x+3}{(x-1)(2x-6)} dx \\ \int \ln(2y) dy &= \int \left(\frac{3}{2x-6} - \frac{1}{x-1} \right) dx \end{aligned}$$

Integrate the LHS using Integration by Parts. Let $u = \ln(2y) \implies du = \frac{1}{y} dy$ and $dv = dy \implies v = y$:

$$\int \ln(2y) dy = y \ln(2y) - \int y \left(\frac{1}{y} \right) dy = y \ln(2y) - y$$

Integrate the RHS directly:

$$\int \left(\frac{3}{2x-6} - \frac{1}{x-1} \right) dx = \frac{3}{2} \ln|2x-6| - \ln|x-1| + C$$

Equate both sides:

$$y \ln(2y) - y = \frac{3}{2} \ln|2x-6| - \ln|x-1| + C$$

Apply the boundary condition ($x = 4, y = 1$):

$$\begin{aligned} 1 \cdot \ln(2) - 1 &= \frac{3}{2} \ln(2(4)-6) - \ln(4-1) + C \\ \ln 2 - 1 &= \frac{3}{2} \ln 2 - \ln 3 + C \\ C &= -\frac{1}{2} \ln 2 + \ln 3 - 1 \end{aligned}$$

Substitute C back into the general solution:

$$y \ln(2y) - y = \frac{3}{2} \ln(2x-6) - \ln(x-1) - \frac{1}{2} \ln 2 + \ln 3 - 1$$