1 (a) Find the coefficient of x^2 in the expansion of $\dfrac{2+3x}{(2+x)^3}$. [4]

$(2+3x)(2+x)^{-3}$

$(2+3x)\,2^{-3}\left(1+\frac{x}{2}\right)^{-3}$

$(2+3x) \times \frac{1}{8}\left[1 + \left(\frac{x}{2}\right)(-3) + \frac{(-3)(-4)\left(\frac{x}{2}\right)^2}{2!}\right]$

$(2+3x) \cdot \frac{1}{8}\left(1 - \frac{3x}{2} + \frac{3}{2}x^2\right)$

$\frac{1}{8}(2+3x)\left(1 - \frac{3x}{2} + \frac{3}{2}x^2\right)$

$\frac{1}{8}(2)\left(\frac{3}{2}x^2\right) + \frac{1}{8}(3x)\left(-\frac{3x}{2}\right)$

$\frac{3}{8}x^2 - \frac{9}{16}x^2$

$= -3/16\, x^2 \quad \therefore$ coef of $x^2 = -3/16$.

(b) State the set of values of x for which the expansion is valid. [1]

$\left|\frac{x}{2}\right| < 1 \quad \therefore \quad |x| < 2$

OR $-2 < x < 2$.

2 A curve has equation $3x\ln y = x^2 - 4y^3$ for $x > 0$.

Find the gradient of the curve at the point where $y = 1$. [4]

$3x\ln y = x^2 - 4y^3$ dif wrt x.

$(\ln y)(3) + 3x\left(\frac{1}{y}\right)\frac{dy}{dx} = 2x - 12y^2\left(\frac{dy}{dx}\right)$

when $y = 1$; $3x\ln(1) = x^2 - 4$

$x^2 = 4$, $x = 2$ $(x > 0)$

$3\ln(1) + 3(2)(1)\frac{dy}{dx} = 2(2) - 12(1)^2\frac{dy}{dx}$.

$6\frac{dy}{dx} = 4 - 12\frac{dy}{dx}$.

$18\frac{dy}{dx} = 4$

$\frac{dy}{dx} = \frac{4}{18} = \frac{2}{9}$

3 The variables x and y satisfy the equation $yx^a = k$, where a and k are constants.

(a) Show that the graph of $\ln y$ against $\ln x$ is a straight line. [2]

$yx^a = k$ take ln.

$\ln(yx^a) = \ln k$.

$\ln y + a\ln x = \ln k$.

$\ln y = -a\ln x + \ln k$.

$Y = MX + C$

$\therefore$ gradient $= -a$

y intercept $= \ln k$.

(b) When $x = 5$, $y = 0.72$ and when $x = 2.3$, $y = 5$.

Find the values of a and k. Give your answers correct to 2 significant figures. [3]

gradient $= \Delta\ln y / \Delta\ln x$

$-a = \dfrac{\ln(5) - \ln(0.72)}{\underbrace{\ln(2.3) - \ln(5)}_{*}}$

$\therefore a = -*$

$= 2.5$.

using either set of points.

$\ln(0.72) = -2.5\ln(5) + \ln k$.

$\ln(0.72) + 2.5\ln(5) = \ln k$.

$\ln[0.72 \cdot 5^{2.5}] = \ln k$.

$k = 0.72(5)^{2.5} = 40$.

4 **(a)** Solve the inequality $|x+4| > |3x-1|$. [3]

^{2} bs.

$(x+4)^2 > (3x-1)^2$

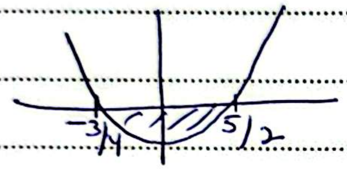
$x^2+8x+16 > 9x^2-6x+1$

$-8x^2+14x+15 > 0$

$8x^2-14x-15 < 0$

$(2x-5)(4x+3) < 0$

CV $\quad x = 5/2 \qquad x = -3/4$.



$-\frac{3}{4} < x < 5/2$

(b) Hence, find the largest integer n for which $|3^{0.05n}+4| > |3^{1+0.05n}-1|$. [2]

$|3^{0.05n}+4| > |3.3^{0.05n}-1|$

comparing to (a) $\qquad x = 3^{0.05n}$

$-\frac{3}{4} < 3^{0.05n} < 5/2$.

$3^{0.05n} < 5/2$, $\quad 0.05n \lg(3) < \lg(2.5)$

$n < \lg(2.5)/0.05\lg3$

$n < 16.68$

$n = 16$.

5 A curve has equation $y = \dfrac{\sin 2x}{3 + \cos x}$ for $0 < x < \pi$.

(a) Express $\dfrac{dy}{dx}$ as a simplified fraction in terms of $\cos x$. [4]

$y' = \dfrac{(3+\cos x)(2\cos 2x) - (\sin 2x)(-\sin x)}{(3+\cos x)^2}$

NUM $= 6\cos 2x + 2\cos 2x \cos x + \sin x(2\sin x \cos x)$

$6(2\cos^2 x - 1) + 2(2\cos^2 x - 1)\cos x + 2(1-\cos^2 x)(\cos x)$

$= 12\cos^2 x - 6 + 4\cos^3 x - 2\cos x + 2\cos x - 2\cos^3 x$

$= 2\cos^3 x - 12\cos^2 x - 6$

$y' \Rightarrow y' = \dfrac{2(\cos^3 x - 6\cos^2 x - 3)}{(3+\cos x)^2}$

(b) Hence, find an equation of the normal to the curve at the point where $x = \frac{1}{2}\pi$. Give your answer in an exact form. [2]

when $x = \pi/2$; $y = 0$.

$y' = -2/3$ $\therefore$ $m_N = 3/2$.

$y - 0 = 3/2(x - \pi/2)$

$y = \frac{3}{2}x - \frac{3\pi}{4}$

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6 A curve is such that the gradient of the curve at any point is proportional to the y-coordinate at that point. The curve passes through the points $(2, e^2)$ and $(5, e^3)$.

Form and solve a differential equation to find y in terms of x. [7]

$\frac{dy}{dx} \propto y \Rightarrow \frac{dy}{dx} = ky$.

$\frac{dy}{y} = k\,dx$ $\int$ bs wrt x.

$\ln y = kx + c$. use given points.

$\ln e^2 = 2k + c$

$2 = 2k + c \to (1)$.

$\ln e^3 = 5k + c$ $(1) - (2) \Rightarrow -1 = -3k$

$3 = 5k + c \to (2)$. $\therefore k = 1/3$.

$2 = 2(1/3) + c \therefore$ $c = 4/3$.

$\ln y = \frac{1}{3}x + \frac{4}{3}$

$\ln y = \frac{x+4}{3}$

$\therefore y = e^{\left(\frac{x+4}{3}\right)}$

7 It is given that $\frac{2+ai}{a-3i} = \lambda(1-5i)$, where a and λ are real constants.

(a) Show that $a^2 - 5a + 6 = 0$. **[4]**

$$\frac{2+ai}{a-3i} \times \frac{a+3i}{a+3i}$$

$$= \frac{2(a+3i)+ai(a+3i)}{a^2-(3i)^2} = \frac{2a+6i+a^2i+3ai^2}{a^2+9}$$

$$= \frac{a^2i+6i-a}{a^2+9} = \lambda - 5\lambda i$$

$$= \frac{-a}{a^2+9} + \frac{i(a^2+6)}{a^2+9} = \lambda - 5\lambda i$$

equate real & imaginary parts.

$$\frac{-a}{a^2+9} = \lambda \rightarrow (1) \qquad \frac{a^2+6}{a^2+9} = -5\lambda \rightarrow (2).$$

$$(1)/(2) = \frac{\left(\frac{-a}{a^2+9}\right)}{\left(\frac{a^2+6}{a^2+9}\right)} = \frac{\lambda}{-5\lambda}$$

$$= \frac{-a}{a^2+6} = \frac{-1}{5} \Rightarrow -5a = -a^2-6$$

$$a^2-5a+6=0$$

$\not{0}$.

(b) Hence, find the possible values of a and the corresponding values of λ. [3]

$a^2 - 5a + 6 = 0$

$(a-3)(a-2) = 0$

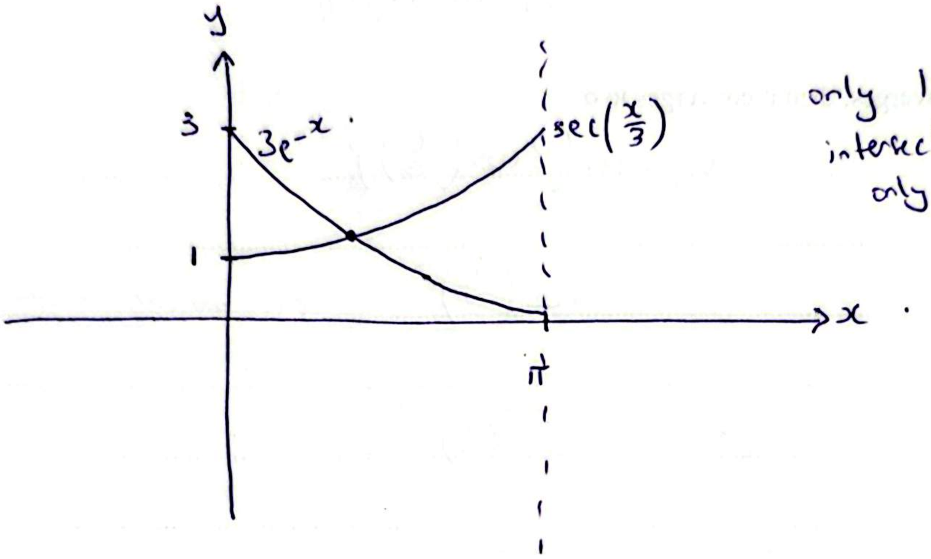
$\therefore a = 3$ or $a = 2$

from real part: $\frac{-a}{a^2+9} = \lambda$

$\therefore \lambda = \frac{-3}{9+9} = \frac{-3}{18} = -1/6$

or $\lambda = \frac{-2}{4+9} = \frac{-2}{13}$

8 **(a)** By sketching a suitable pair of graphs, show that the equation $\sec\left(\frac{1}{3}x\right) = 3e^{-x}$ has exactly one root in the interval $0 < x < \pi$. [2]



only 1 pt of intersection ∴ only 1 root.

(b) The root is denoted by α. Show that $1 < \alpha < 1.1$. [2]

$p(x) = \sec\left(\frac{x}{3}\right) - 3e^{-x} = 0$.

$x = 1$, $p(x) = -0.0454$

$x = 1.1$, $p(x) = 0.0726$

sign change occurs between $x = 1$ & 1.1 ∴ root must lie in this interval.

(c) Show that, if a sequence of positive values given by the formula

$$x_{n+1} = \ln\left(3\cos\left(\tfrac{1}{3}x_n\right)\right)$$

converges, then it converges to α. [2]

$x = \ln\left(3\cos\left(\frac{x}{3}\right)\right)$

$e^{x} = 3\cos\left(\frac{x}{3}\right)$ reciprocate both sides.

$\frac{1}{e^{x}} = \frac{1}{3\cos\left(\frac{x}{3}\right)}$

$3/e^{x} = \frac{1}{\cos\left(\frac{x}{3}\right)}$

$3e^{-x} = \sec\left(\frac{x}{3}\right)$ ✓

(d) Use this iterative formula, with initial value $x_1 = 1$, to calculate α correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

$x_1 = 1$

$x_2 = 1.04200$

$x_3 = 1.03704$

$x_4 = 1.03764$

$x_5 = 1.03756$

$x_6 = 1.03757$ $\therefore$ $\alpha \approx 1.037$

$x_7 = 1.03757$

9 (a) Express $5\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. State the exact value of R, and give the value of α in the form $\tan^{-1}(p)$, where p is a rational number. [3]

$5\cos\theta - 3\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$R\cos\alpha = 5$ & $R\sin\alpha = 3$

$R = \sqrt{5^2 + 3^2} = \sqrt{34}$.

$\tan\alpha = 3/5 \quad \therefore \quad \alpha = \tan^{-1}(3/5)$

(b) Hence, show that, when α has the value found in **9(a)**,

$$\int_0^{\alpha} \frac{68}{(5\cos\theta - 3\sin\theta)^2}\,d\theta = \frac{51}{20}.$$

[4]

$5\cos\theta - 3\sin\theta = \sqrt{34}\cos\left(\theta + \frac{3}{5}\right)$

$$\int_0^{\tan^{-1}(\frac{3}{5})} \frac{68}{34\cos^2(\theta+\frac{3}{5})}\,d\theta = \frac{68}{34}\int_0^{\tan^{-1}(\frac{3}{5})} \sec^2\left(\theta+\frac{3}{5}\right)d\theta.$$

$$= \frac{68}{34}\left[\tan\left(\theta+\frac{3}{5}\right)\right]_0^{\tan^{-1}(\frac{3}{5})}$$

~~Recall, $\alpha =$~~

$$= \frac{68}{34}\left[\tan(\theta+\alpha)\right]_0^{\alpha}.$$

$$= \frac{68}{34}\left[\tan 2\alpha - \tan\alpha\right]$$

$\alpha = \tan^{-1}\left(\frac{3}{5}\right)$

$\tan\alpha = 3/5.$

$$\frac{2\tan\alpha}{1-\tan^2\alpha} - \tan\alpha$$

$$\frac{68}{34}\left[\frac{2(3/5)}{1-\frac{9}{25}} - \frac{3}{5}\right]$$

$$\frac{68}{34}\left(\frac{51}{40}\right) = \frac{51}{20}.$$

$\cancel{D}$

10 (a) Use the substitution $x = 2\sin u$ to show that

$$\int_0^1 \frac{3x^3}{\sqrt{4-x^2}}\,dx = \int_a^b 24\sin^3 u\,du,$$

where a and b are exact values to be found. [4]

$x = 2\sin u$

$dx = 2\cos u\,du.$

$1 = 2\sin u \quad \therefore \quad u = \sin^{-1}(1/2) = \pi/6.$

$0 = 2\sin u \quad \therefore \quad u = 0.$

$$\int_0^{\pi/6} \frac{3(8\sin^3 u)}{\sqrt{4-4\sin^2 u}}\,2\cos u\,du.$$

$$= \int_0^{\pi/6} \frac{24\sin^3 u \cdot 2\cos u\,du}{\sqrt{4(1-\sin^2 u)}} = \frac{24\sin^3 u \cdot 2\cos u\,du}{\sqrt{4\cos^2 u}}$$

$$= \frac{24\sin^3 u \cdot \cancel{2\cos u}\,du}{\cancel{2\cos u}}$$

$$= \int_0^{\pi/6} 24\sin^3 u\,du.$$

(b) Hence, find the exact value of $\int_0^1 \frac{3x^3}{\sqrt{4-x^2}}\,dx$. $= \int_0^{\frac{\pi}{6}} 24\sin^3 u \, du$. [4]

$\sin^3 u = \sin^2 u(\sin u)$

$= (1-\cos^2 u)(\sin u)$

$= 24\int_0^{\frac{\pi}{6}} (\sin u - \cos^2 u \sin u)\, du$

$= -\cos u + \frac{1}{3}\cos^3 u \Big|_0^{\pi/6} \times 24$

$= \left[\frac{-\sqrt{3}}{2} + \frac{\sqrt{3}}{8} - \left(-1 + \frac{1}{3}\right)\right] \times 24$

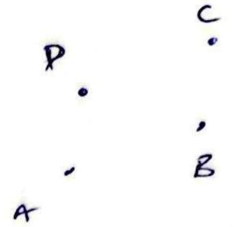
$= \frac{16-9\sqrt{3}}{24} \times 24$

$= 16 - 9\sqrt{3}$

Parallelogram – opposite sides equal & parallel.

11 Relative to a fixed point, O, the points A, B, C and D have position vectors

$$\overrightarrow{OA} = \begin{pmatrix}2\\3\\5\end{pmatrix},\quad \overrightarrow{OB} = \begin{pmatrix}3\\1\\4\end{pmatrix},\quad \overrightarrow{OC} = \begin{pmatrix}-1\\0\\-1\end{pmatrix} \quad\text{and}\quad \overrightarrow{OD} = \begin{pmatrix}-3\\4\\1\end{pmatrix}.$$



(a) Show that $ABCD$ is a trapezium, and **not** a parallelogram. [4]

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \begin{pmatrix}-2\\-3\\-5\end{pmatrix} + \begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}1\\-2\\-1\end{pmatrix}$

$\overrightarrow{DC} = \overrightarrow{DO} + \overrightarrow{OC} = \begin{pmatrix}3\\-4\\-1\end{pmatrix} + \begin{pmatrix}-1\\0\\-1\end{pmatrix} = \cancel{\begin{pmatrix}-6\\-3\\-3\end{pmatrix}}\ \begin{pmatrix}2\\-4\\-2\end{pmatrix}$

$\overrightarrow{AD} = \overrightarrow{AO} + \overrightarrow{OD} = \begin{pmatrix}-2\\-3\\-5\end{pmatrix} + \begin{pmatrix}-3\\4\\1\end{pmatrix} = \begin{pmatrix}-5\\1\\-4\end{pmatrix}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = \begin{pmatrix}-3\\-1\\-4\end{pmatrix} + \begin{pmatrix}-1\\0\\-1\end{pmatrix} = \begin{pmatrix}-4\\-1\\-5\end{pmatrix}$

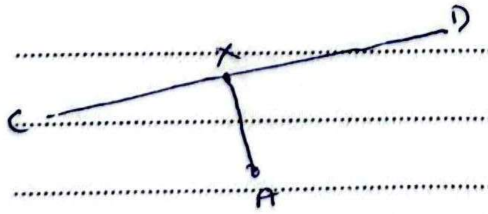
$2\overrightarrow{AB} = \overrightarrow{DC}$ $\therefore$ parallel. $\overrightarrow{AD}$ is NOT a scalar multiple of $\overrightarrow{BC}$ $\therefore$ NOT parallel.

$|\overrightarrow{AB}| = \sqrt{6}$ $\quad |\overrightarrow{DC}| = 2\sqrt{6}$ $\therefore$ opposite sides not equal.

Only 1 pair of parallel sides $\therefore$ trapezium.

(b) Show that the distance from A to the line through C and D is $\frac{9}{2}\sqrt{2}$. [5]

$\overrightarrow{CD} = \begin{pmatrix}-2\\4\\2\end{pmatrix}$ $\quad$ vector eqn $\Rightarrow r = \begin{pmatrix}-1\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}$



general point on CD, say $\overrightarrow{OX} = \begin{pmatrix}-1-\lambda\\2\lambda\\-1+\lambda\end{pmatrix}$

$\overrightarrow{AX} = \overrightarrow{AO} + \overrightarrow{OX} = \begin{pmatrix}-2\\-3\\-5\end{pmatrix} + \begin{pmatrix}-1-\lambda\\2\lambda\\-1+\lambda\end{pmatrix} = \begin{pmatrix}-3-\lambda\\-3+2\lambda\\-6+\lambda\end{pmatrix}$

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$\overrightarrow{AX} \cdot \text{direction of } CO = 0$.

$$\begin{pmatrix} -3-\lambda \\ -3+2\lambda \\ -6+\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = 0 \Rightarrow 3+\lambda-6+4\lambda-6+\lambda=0.$$

$$-9+6\lambda=0$$

$$\lambda = -9/-6 = +3/2.$$

$$|\overrightarrow{AX}| = \left| \begin{matrix} -4.5 \\ 0 \\ -4.5 \end{matrix} \right|$$

$$= \sqrt{4.5^2+4.5^2} = \frac{9\sqrt{2}}{2}$$

$$= \frac{9}{2}\sqrt{2}.$$

(c) Find the exact area of *ABCD*. Give your answer in simplified form. [3]

$|AB| = \sqrt{6}$

$|DC| = 2\sqrt{6}$

$\perp$ distance $= \frac{9}{2}\sqrt{2}$

Area $=$ ~~$\frac{1}{2}(AB)(CD)$~~ $\frac{1}{2}(|\overrightarrow{AB}|+|\overrightarrow{DC}|)\frac{9}{2}\sqrt{2}$

$$= \frac{1}{2}\left(\sqrt{6}+2\sqrt{6}\right)\frac{9}{2}\sqrt{2}$$

$$= \frac{9}{4}\sqrt{2}\left(3\sqrt{6}\right) = \frac{27\sqrt{12}}{4}$$

$$= \frac{27\sqrt{4}\sqrt{3}}{4}$$

$$= 27\sqrt{3}/2.$$